Lecture 10

ORDINARY DIFFERENTIAL EQUATIONS

DEFINITION

An equation containing the differentiation of one or more dependent variables, with respect to one or more independent variables

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

1. TYPE
2. ORDER
3. LINEARITY

TYPE

ORDINARY

Involve only **ONE** independent variable

$$y' - 3y = e^{2x}$$

$$y'' + 2y' - 3y = 0$$

PARTIAL

Involve **TWO or MORE** independent variable

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

ORDER

The *order* of differential equation is the order of the highest differentiation

$$y'' + 2y' - 3y = 0$$

second order first order

LINEARITY

The differential equation of order "n" is said to be linear, if the function

$$F\left(x, y', y'', \ldots, y^{(n)}\right) = 0 \quad \text{or} \quad F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

is linear in the variables y, y',…,$y^{(n)}$

There are no multiplications of dependent variables and their differentiations.

$$y'' + 2x^3 y' - 3y = 0$$ – **linear DE**

$$y'' + 2y \cdot y' - 3y = 0$$ – **nonlinear DE**

LINEAR ODE

HOMOGENOUS

$$p_n(x)y^{(n)}(x)+p_{n-1}(x)y^{(n-1)}(x)+\ldots+p_2(x)y''+p_1(x)y'+p_0(x)y(x)=0$$

$$(1+x^3)y^{(5)}-y^{(4)}(x)+xy'+3y=0$$ where

$$p_5=1+x^3,\ p_4=-1,\ p_3=0,\ p_2=0,\ p_1=x,\ p_0=3$$

IMPORTANT PROPERTY OF LINEAR DE

If **$y_1(x)$** is first solution and **$y_2(x)$** is the second solution, then the function **$y_1(x) + a\, y_2(x)$** is **solution**, too (a is real number)

NON-HOMOGENOUS

$$y'+y+2=0$$

ORIGIN OF DE

- PHYSICS (and all other natural sciences)

- ECONOMICS

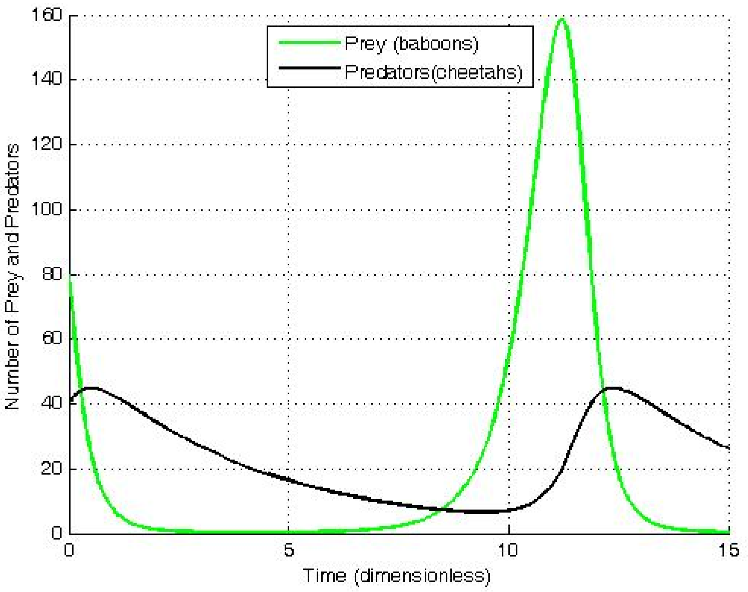
- SOCIOLOGY

EXAMPLE FROM BIOLOGY

PREDATOR – PREY MODEL (Lotka – Volterra model)

$$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = \delta xy - \gamma x$$

where
- x is the number of prey (e.g. rabbits)
- y is the number of predator (e.g. foxes)
- dx/dt and dy/dt represent the growth rates of these two population over time t
- α, β, γ, δ are positive real parameters describing the interaction of the species



SOLUTIONS OF ODE

ELEMENTARY SITUATION

$$\frac{dy}{dx} = y'(x) = f(x)$$

SOLUTION by simple integration: $y(x) = \int f(x)\,dx$

- generalization: $y^{(n)}(x) = f(x) \rightarrow y(x) = \underbrace{\int\int\ldots\int}_{n\ \text{times}} f(x)\,dx\,dx\ldots dx$

IMPORTANT: The first solution (the DE of the first order) will contain **ONE** integration constant, because of **ONE** integration. The solution of generalized case will contain **n** constants because of **n** integrations. The final values of the constants are dependent on **boundary conditions.**

EXAMPLE 1

$$y'(x) = x - \sin x$$ boundary condition: $$y(0) = -1$$

Solution

$$y(x) = \int (x - \sin x)\, dx = \frac{x^2}{2} + \cos x + C$$

The constant C is set through boundary condition:

$$y(0) = -1 \rightarrow \frac{0^2}{2} + \cos 0 + C = -1 \Rightarrow C = -2$$

The final solution is:

$$y(x) = \frac{x^2}{2} + \cos x - 2$$

EXAMPLE 2

$$y'''(x) = 1 - x^2$$ no boundary conditions

Solution

$$y(x) = \int\left(\int\left(\int(1 - x^2)\,dx\right)dx\right)dx = -\frac{x^5}{60} + \frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3$$

EXAMPLE 3

$$y''(x) = x^2$$ boundary conditions: $y(0) = 0;\ y'(0) = 1$

Solution

$$y(x) = \int\left(\int x^2 dx\right)dx = \frac{x^4}{12} + C_1 x + C_2$$

$$y(0) = 0 \Rightarrow C_2 = 0; \quad y'(0) = 1 \Rightarrow C_1 = 1$$

SOLUTIONS OF ODE

DE of the 1st order in SEPARABLE FORM

$$y'(x) = f(x) \cdot g(y)$$

SOLUTION

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$ separation

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$ integration

$$y(x) = \cdots$$ explicit form if possible

EXAMPLE 1

$$y'(x) = -y^4$$ boundary condition: $$y(0) = 1$$

Solution

$$\frac{dy}{dx} = -y^4$$

$$\frac{dy}{y^4} = -dx \rightarrow \int \frac{dy}{y^4} = -\int dx$$

$$\frac{1}{-3y^3} = -x + C \Rightarrow y = \frac{1}{\sqrt[3]{3x + C}}$$

The constant C is: $C = 1$

The final solution is: $y(x) = \frac{1}{\sqrt[3]{3x + 1}}$

EXAMPLE 2

$$y'(x) = \frac{y}{y-1}x$$

boundary condition: $y(2) = 2$

Solution

$$\frac{dy}{dx} = \frac{y}{y-1}x$$

$$\frac{y-1}{y}dy = xdx \rightarrow \int \frac{y-1}{y}dy = \int xdx$$

$$y - \ln|y| = \frac{x^2}{2} + C$$

can not be rewritten to explicit form

The constant C is: $2 - \ln|2| = \frac{2^2}{2} + C \Rightarrow C = -\ln 2$

The final solution is: $\frac{x^2}{2} = y - \ln|y| + \ln 2$

SOLUTIONS OF ODE

Linear DE of the 1st order

- **HOMOGENOUS** $y'(x)+p(x)\cdot y(x)=0$

  where p(x) is given function.

- **NON-HOMOGENOUS** $y'(x)+p(x)\cdot y(x)=q(x)$

  where p(x) and q(x) are given functions.

SOLUTION OF HOMOGENOUS CASE

$$\frac{dy}{dx} = -p(x) \cdot y(x)$$

$$\frac{dy}{y(x)} = -p(x)dx$$ separation

$$\int \frac{dy}{y} = -\int p(x)dx$$ integration

$$\ln\left|\frac{y(x)}{C}\right| = -\int p(x)dx \Rightarrow$$

$$y(x) = C\exp\left(-\int p(x)dx\right)$$ final solution

EXAMPLE 1

$$y'(x) = y \cdot \cos x$$ boundary condition: $$y(0) = 1$$

Solution

$$\frac{dy}{dx} = y \cos x$$

$$\frac{dy}{y} = \cos x \, dx \rightarrow \int \frac{dy}{y} = \int \cos x \, dx$$

$$\ln\left|\frac{y}{C}\right| = \sin x \Rightarrow y = Ce^{\sin x}$$

The constant C is: $C = 1$

The final solution is: $y(x) = e^{\sin x}$

EXAMPLE 2

$$y'(x) - xy(x) = 0$$ boundary condition: $$y(1) = 1$$

Solution

$$\frac{dy}{dx} = yx$$

$$\frac{dy}{y} = x\,dx \rightarrow \int \frac{dy}{y} = \int x dx$$

$$\ln\left|\frac{y}{C}\right| = \frac{x^2}{2} \Rightarrow y = Ce^{\frac{x^2}{2}}$$

The constant C is: $C = e^{-\frac{1}{2}}$

The final solution is: $y(x) = y = e^{\frac{x^2}{2} - \frac{1}{2}}$

SOLUTION OF NON-HOMOGENOUS CASE

$$y'(x) + p(x) \cdot y(x) = q(x)$$

TWO METHODS

- substitution
- variation of the constant

1. step: RHS is set to be 0 $\quad y'(x) + p(x) \cdot y(x) = 0$

2. step: solving of homogenous case through separation $\quad y(x) = C \exp\left(-\int p(x)\,dx\right)$

3. step: introducing constant C as the function of “x” in (2) $\quad C \to C(x)$

4. step: differentiation of (2), where C is C(x)

5. step: put “y(x)” and “y’(x)” in original DE and calculate the C(x)

6. step: put C(x) in y(x) and obtain **final solution**

EXAMPLE 1

$$y'(x) + y \cdot \cos x = \sin 2x$$

Solution

1. RHS = 0 $y'(x) + y \cdot \cos x = 0$

2. separation $y = Ce^{-\sin x}$

3. constant C as the function $y = C(x)e^{-\sin x}$

4. differentiation $y' = C'e^{-\sin x} - Ce^{-\sin x}\cos x$

5. calculating of C $C'e^{-\sin x} - Ce^{-\sin x}\cos x + Ce^{-\sin x} \cdot \cos x = \sin 2x$

$$C'e^{-\sin x} = \sin 2x$$

$$C' = e^{\sin x}\sin 2x \to C(x) = \int e^{\sin x}\sin 2x\, dx$$

$$C(x) = 2e^{\sin x}(\sin x - 1) + K$$

6. final solution $y = \left(2e^{\sin x}(\sin x - 1) + K\right)e^{-\sin x} = 2(\sin x - 1) + Ke^{-\sin x}$

EXAMPLE 2

$$\cos x - y - y'(x) = 0$$

Solution

1. RHS = 0 $y' + y = 0$

2. separation $y = Ce^{-x}$

3. constant C as the function $y = C(x)e^{-x}$

4. differentiation $y' = C'e^{-x} - Ce^{-x}$

5. calculating of C

$$C'e^{-x} = \cos x$$

$$C = \int e^{x} \cos x \, dx$$

$$C(x) = e^{x} \frac{\sin x + \cos x}{2} + K$$

6. final solution $$y = \frac{\sin x + \cos x}{2} + Ke^{-x}$$

SOLUTIONS OF ODE

Linear DE of the 2nd order

- **HOMOGENOUS** $$y''(x)+p\cdot y'(x)+q\cdot y(x)=0$$
  where p and q are real constants

- **NON-HOMOGENOUS** $$y''(x)+p\cdot y'(x)+q\cdot y(x)=f(x)$$
  where p and q are real constants

SOLUTION OF HOMOGENOUS CASE

IMPORTANT
There **MUST** be **two** **linearly independent** solutions

Solution is searched in form: $y(x) = e^{\lambda x}$

1. step: calculation of λ $\lambda^2 + p\lambda + q = 0$ **(characteristic function)**

2. step: choosing of proper solution form according to discriminant D of previous equation

$$1.\ \text{if} \quad D = p^2 - 4q > 0 \quad \rightarrow \quad y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$2.\ \text{if} \quad D = p^2 - 4q < 0 \quad \rightarrow \quad y(x) = e^{-\frac{p}{2}x}\left[C_1 \cos(\omega x) + C_2 \sin(\omega x)\right]$$

$$\text{where} \quad \omega = \sqrt{q - \frac{p^2}{4}}$$

$$3.\ \text{if} \quad D = p^2 - 4q = 0 \quad \rightarrow \quad y(x) = C_1 e^{-\frac{p}{2}x} + C_2 x e^{-\frac{p}{2}x}$$

SPECIAL CASE $p = 0 \wedge q = 1 \rightarrow y''(x) + y(x) = 0$

SOLUTION OF SPECIAL CASE $y(x) = C_1 \cos x + C_2 \sin x$

EXAMPLE 1

$$y''(x) - y'(x) - 2y(x) = 0, \qquad y(0) = 1,\ y'(0) = 0$$

Solution

1. expected solution $y(x) = e^{\lambda x}$

2. characteristic function $\lambda^2 - \lambda - 2 = 0$

3. λ $\lambda_1 = -1, \quad \lambda_2 = 2$

4. general solution $y(x) = C_1 e^{-x} + C_2 e^{2x}$

5. boundary conditions

$$\left.\begin{aligned} y(0) = 1 &\rightarrow C_1 + C_2 = 1 \\ y'(0) = 0 &\rightarrow -C_1 + 2C_2 = 0 \end{aligned}\right\rangle \Rightarrow \left\langle \begin{aligned} C_1 &= \frac{2}{3} \\ C_2 &= \frac{1}{3} \end{aligned}\right.$$

6. final solution $y(x) = \frac{1}{3}\left(2e^{-x} + e^{2x}\right)$

EXAMPLE 2

$$y''(x) - 2y'(x) + 2y(x) = 0, \qquad y(0) = 1,\ y'(0) = 0$$

Solution

1. expected solution $y(x) = e^{\lambda x}$

2. characteristic function $\lambda^2 - 2\lambda + 2 = 0$

3. λ $D < 0 \rightarrow \lambda_{1,2} \in \mathbb{C} \qquad \omega \ddagger 1$

4. general solution $y(x) = e^{x}\left(C_1 \cos x + C_2 \sin x\right)$

5. boundary conditions

$$\left.\begin{array}{l} y(0) = 1 \rightarrow C_1 = 1 \\ y'(0) = 0 \rightarrow C_1 + C_2 = 0 \end{array}\right\rangle \Rightarrow \left\langle\begin{array}{l} C_1 = 1 \\ C_2 = -1 \end{array}\right.$$

6. final solution $y(x) = e^{x}\left(\cos x - \sin x\right)$

EXAMPLE 3

$$y''(x)+6y'(x)+9y(x)=0, \qquad y(0)=1, y'(0)=1$$

Solution

1. expected solution $y(x)=e^{\lambda x}$

2. characteristic function $\lambda^2+6\lambda+9=0$

3. λ $D=0 \rightarrow \lambda=3$

4. general solution $y(x)=\left(C_1e^{-3x}+C_2xe^{-3x}\right)$

5. boundary conditions $\left.\begin{array}{l} y'(0)=1 \rightarrow -3C_1+C_2=1 \\ y(0)=1 \rightarrow C_1=1 \end{array}\right\rangle \Rightarrow \left\langle \begin{array}{l} C_1=1 \\ C_2=4 \end{array}\right.$

6. final solution $y(x)=e^{-3x}(1+4x)$