Lecture 2: Linear algebra, coordinate systems, vector analysis

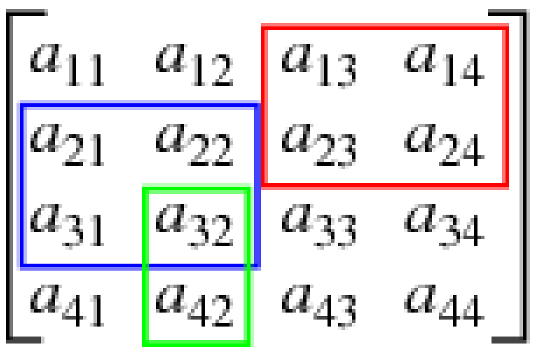
Content:

- addition to previous lecture
- systems of linear equations, methods of solution
- coordinate systems
- vectors, basic definitions and operations

Submatrix:

A submatrix of a matrix is obtained by deleting any collection of rows and/or columns.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \end{bmatrix}$$



submatrices

Used e.g. during determinants evaluation.

Main operations with square matrices (2/3) :

Determinant – evaluation:

The rule of Sarus (or Sarrus' scheme).

For a 3 x 3 matrix it is valid:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} =$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

Another effective way how to compute it:

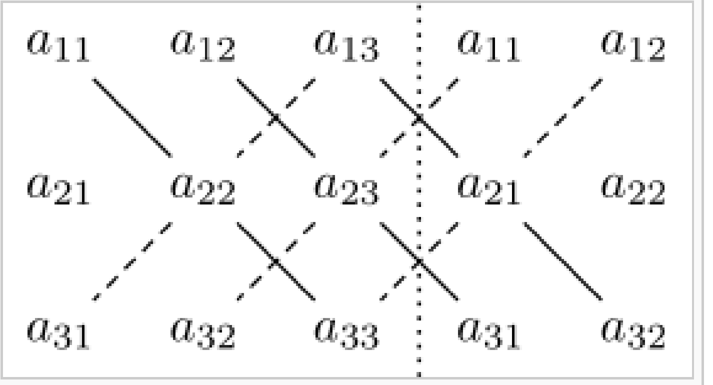
Main operations with square matrices (2/3) :

Determinant – evaluation:

The rule of Sarus (or Sarrus' scheme).

For a 3 x 3 matrix it is valid:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



Sarrus' rule: The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals

$$\det(M) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

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Equations in mathematics

Definition: Equation is an equality containing one or more variables (named as **unknowns**).

Solving the equation consists of determining which values of the variables make the equality true (these values are then called as **solutions**).

Often used symbols for unknowns are: x, y, ...

There exists many types of equations and ways how to classify them. The most important one is based on the power of the unknown in the equation: x, x^2, x^3, ...

linear equation: $3x+5=4x-12$

quadratic equation: $x^2-x-2=0$

cubic equation: $x^3-6x^2+11x-6=0$

(next names for higher polynomials: quartic quintic sextic septic,...)

In the next part of the lecture we will focuse on the properties of linear equations (and systems of linear equations).

linear equations:

Comment: Another important property of linear equations is that each term is either a constant or the product of a constant and (the first power of) a single variable.

Linear equations can have not only one unknown, there can occurre several ones. They are often named as x, y,... or x_1, x_2,...

Since terms of linear equations cannot contain products of distinct or equal variables, nor any power (other than 1) or other function of a variable, equations involving terms such as xy, x^2, $y^{1/3}$, and sin(x) are nonlinear.

examples:

$$6x + 5y = -15$$

$$2x - 5xy = 15 + x$$

$$2tg(x) - 5x = 9$$

linear equations – one variable:

A linear equation in one unknown x may always be rewritten in the usual form:

$$ax = b$$

if a ≠ 0, there is a unique solution:

$$x = b/a$$

linear equations – two variables:

A linear equation with unknowns x and y is usually written in following forms:

$$y = px + q \quad \text{or} \quad Ax + By = C$$

where p and q (A, B, C) designate constants.

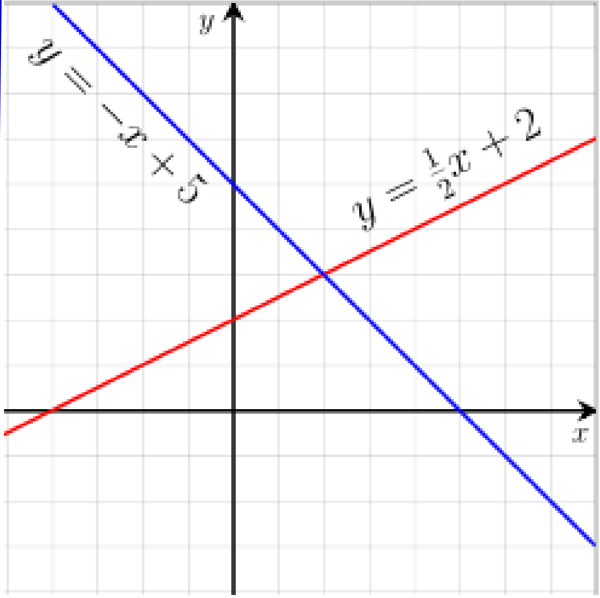
The origin of the name "linear" comes from the fact that the solution of such an equation forms a straight line in the plane. In this particular equation, the constant p determines the **slope** or **gradient** of that line, and the constant term q determines the point at which the line crosses the y-axis, otherwise known as the **y-intercept**.

example:

$$6x + 3 = 3x - 12$$

$$3x = -15$$

$$x = -15/3$$



linear equations – two variables:

Solution methods:

1. substitution method
2. addition method

example:

$$2x + 3y = 6$$
$$4x + 9y = 15$$

solution: y=1, x = 3/2

Comment: In the case of mathematical equations and/or systems of equations we have to check the obtained solution and perform a proof.

linear equations – two variables:

Beside of the general (standard) form, there are several forms of its presentation. Among them the matrix form is very important:

Matrix form

Using the order of the standard form

$$Ax + By = C,$$

one can rewrite the equation in matrix form:

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C \end{pmatrix}.$$

Further, this representation extends to systems of linear equations.

$$A_1x + B_1y = C_1,$$
$$A_2x + B_2y = C_2,$$

becomes:

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

This will be generalized in systems with larger number of variables

Systems of linear equations:

A general system of m linear equations with n unknowns can be written:

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Here x_1, x_2, ...x_n are the unknowns, a_{11},a_{12}, ... a_{mn} are the coefficients of the system, and b_1,b_2, ,,,b_m are the constant terms.

The matrix form **Ax** = **b** is very efficient here:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where **A** is an m $\times$ n matrix (matrix of the system), **x** is a column vector with n entries (solution vector) and **b** is a column vector with m entries (so called right-hand side vector).

Systems of linear equations:

Even a more compact writing form is used (so called augmented matrix):

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

A **solution** to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied.

But when such kind of a system has a solution?

1. If N = M then there are as many equations as unknowns, and there is a good chance of solving for a unique solution.
2. If N > M then the number of equations is unsufficient (fewer equations than unknowns – so called underdetermined system) and there is in general no solution (but this case can be treated in a special way).
3. If N < M then the number of equations "is too large", we call such a system as overdetermined. We have also here a problem, but this situation can be solved by means of so called LSQ-method.

Systems of linear equations:

2. case (N > M), undertermined system, example:

$$x + y + z = 1$$
$$x + y + z = 0$$

3. case (N < M), overdetermined system, example:

$$2x_1 + x_2 = -1$$
$$-3x_1 + x_2 = -2$$
$$-x_1 + x_2 = 1$$

Systems of linear equations - solution:

There exist several methods for the solution of linear equations systems:

- Cramer's rule
- elimination methods

Cramer's rule:

Solution of a system of linear equations **Ax** = **b** is given (where **A** is a square n × n matrix and **A** has a nonzero determinant):

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

where $\mathbf{A}_i$ is the matrix formed by replacing the i-th column of **A** by the column vector **b**.

It is very important that det(**A**) is nonzero → so called regular solution.

In the opposite situation (det(**A**)=0) we have so called singular solution.

Cramer's rule:

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

example (2×2):

Consider the linear system

$$\begin{cases} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Assume $a_1b_2 - b_1a_2$ nonzero. Then, with help of determinants x and y can be found with Cramer's rule as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}$$

Cramer's rule:

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

example (3×3):

The rules for 3 × 3 matrices are similar. Given

$$\begin{cases} a_1x + b_1y + c_1z & = d_1 \\ a_2x + b_2y + c_2z & = d_2 \\ a_3x + b_3y + c_3z & = d_3 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Then the values of x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Elimination methods:

Idea of this approach is based in the step-by-step elimination of variables to obtain so called upper triangular matrix (which can be the solved by means of back-substitution).

Instead of the standard form: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ & \dots & & \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_M \end{bmatrix}$$

we try to obtain the following form (here an example for 4×4):

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & a'_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a'_{33} & a'_{34} \\ 0 & 0 & 0 & a'_{44} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \end{bmatrix}$$

Such an procedure is termed as Gaussian elimination.

Elimination methods:

But how to do it?

We can do this by means of three types of elementary row operations:
1: Swapping the positions of two rows.
2: Multiplying a row by a nonzero scalar.
3: Adding to one row a scalar multiple (linear combination) of another.
This procedures are often named as row reductions.

Example: An augmented matrix of a system

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

1.step: first row is multiplied by 3 and subtracted from the second one

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

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2.step: first row is multiplied by 2 and subtracted from the third one

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

3.step: second row is divided by -4

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

4.step: second row is multiplied by 2 and added to the third one

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Comment: automated algorithm for row reduction is of course more complicated.

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Received matrix is in the form of the needed upper triangular form:

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a'_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

And we can obtain the solutions x_1, x_2 and x_3 by backsubstitution:

$$x_3 = b'_3 / a'_{33}$$

$$x_2 = \frac{1}{a'_{22}} [b'_2 - x_3 a'_{23}]$$

$$x_i = \frac{1}{a'_{ii}} \left[b'_i - \sum_{j=i+1}^{N} a'_{ij} x_j \right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a'_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

In this case: $x_3 = b'_3/a'_{33} = 2/1 = 2$

$a'_{22}x_2 + a'_{23}x_3 = b'_2$

$x_2 + (-3)x_3 = 2 \qquad (x_3 = 2)$

$x_2 - 6 = 2$

$x_2 = 8$

$a'_{11}x_1 + a'_{12}x_2 + a'_{13}x_3 = b'_1$

$x_1 + 3x_2 + (-2)x_3 = 5 \qquad (x_3 = 2,\ x_2 = 8)$

$x_1 + 3\cdot 8 + (-2)\cdot 2 = 5$

$x_1 + 24 - 4 = 5$

$x_1 = -15$

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Coordinate systems – introduction:

In geometry, a coordinate system is a system which uses one or more numbers, or coordinates, to uniquely determine the position of a point or other geometric element in a defined space (Euclidean space).

Predominantly used space in natural sciences
is the 3D Euclidean space, especially the real coordinate space (R^n) (named after the Ancient Greek mathematician Euclid of Alexandria)

There exist several coordinate systems, each of them can be used in different situations in an effective way.

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Predominantly used space in natural sciences is the 3D Euclidean space, especially the real coordinate space ($\mathbf{R}^n$) (named after the Ancient Greek mathematician Euclid of Alexandria)

There exist several coordinate systems, each of them can be used in different situations in an effective way.

Cartesian coordinate system is the mostly used one.

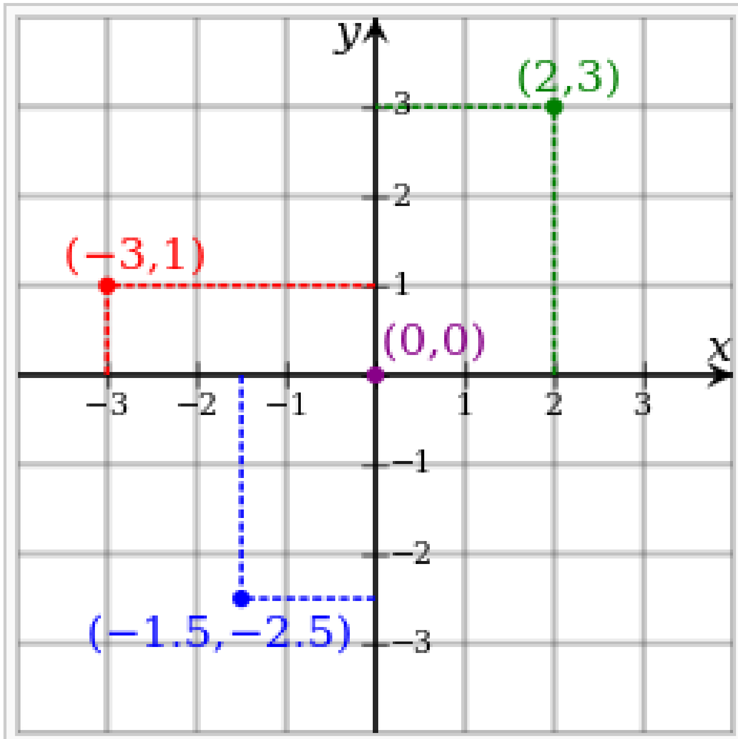
Coordinate systems – Cartesian coordinate system:

Cartesian coordinate system

A coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances from the point to two fixed perpendicular directed lines, measured in the same unit of length.

Comment: this definition is valid for a plane – 2D Cartesian system

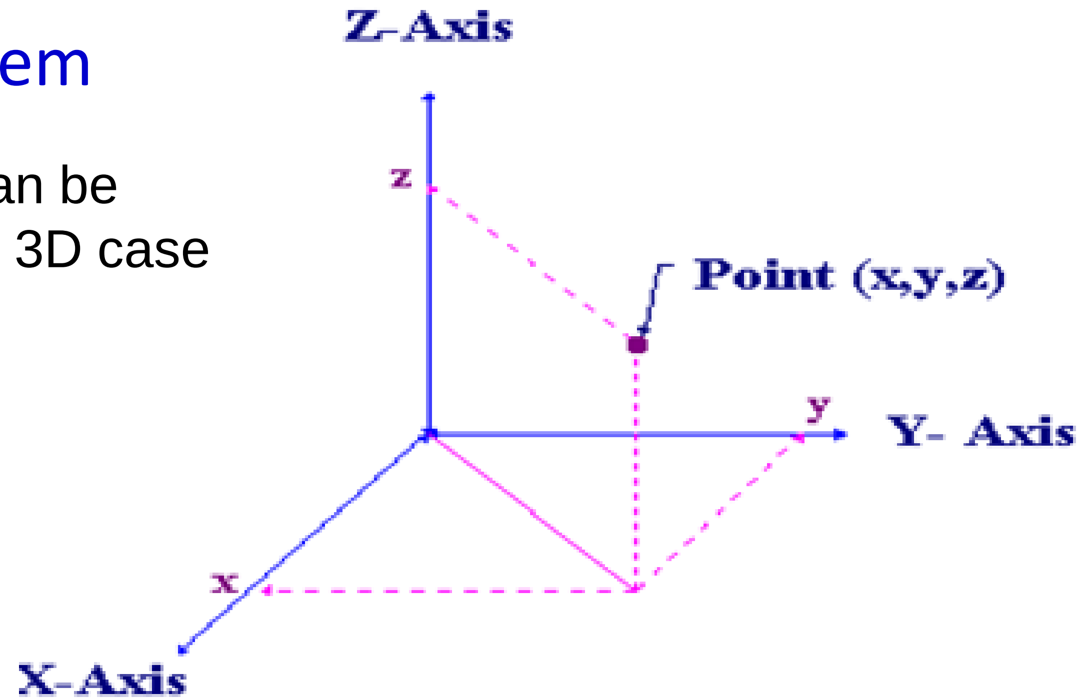
The invention of Cartesian coordinates in the 17th century by René Descartes (Latinized name: Cartesius).



Coordinate systems – Cartesian coordinate system:

Cartesian coordinate system

its definition can be expanded to a 3D case



Elementary vectors: $\vec{\mathrm{i}}, \vec{\mathrm{j}}, \vec{\mathrm{k}}$ (pointing in the direction of each coordinate axis)

Distance between two points

The Euclidean distance between two points of the plane with Cartesian coordinates (x_1, y_1) and (x_2, y_2) is

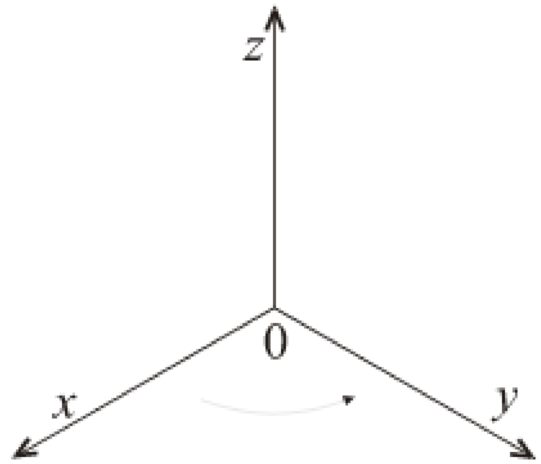
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is the Cartesian version of Pythagoras's theorem. In three-dimensional space, the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

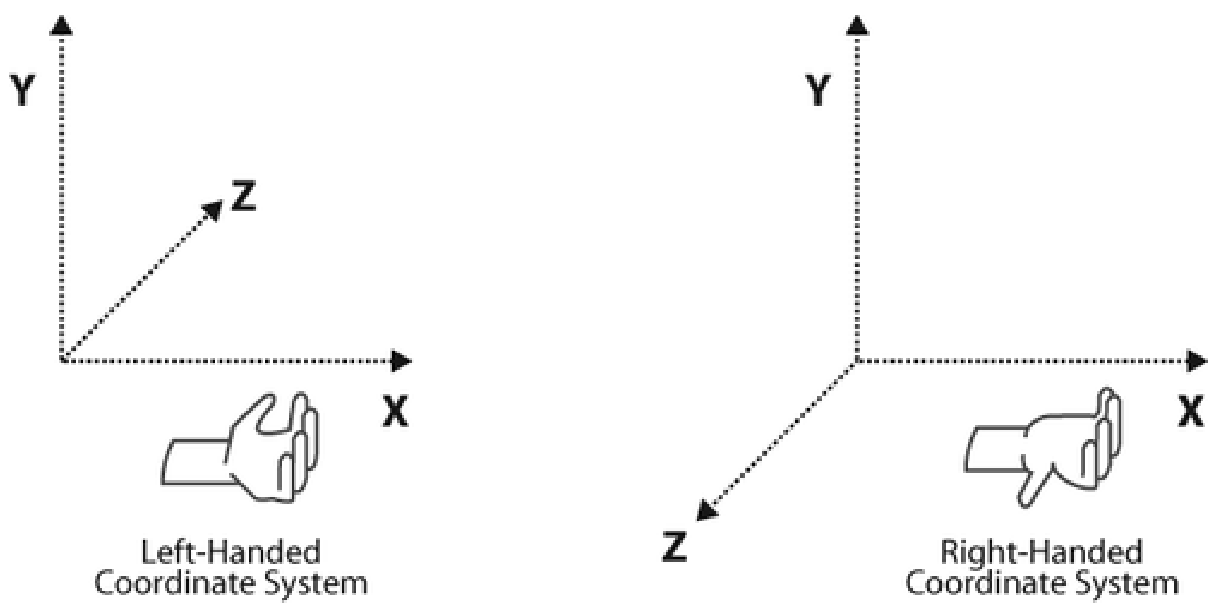
which can be obtained by two consecutive applications of Pythagoras' theorem.

Coordinate systems – Cartesian coordinate system:



In physics and in many technical applications we use in the majority the so called right-handed coordinate system (in 3D space).

Left-handed versus right-handed coordinate systems

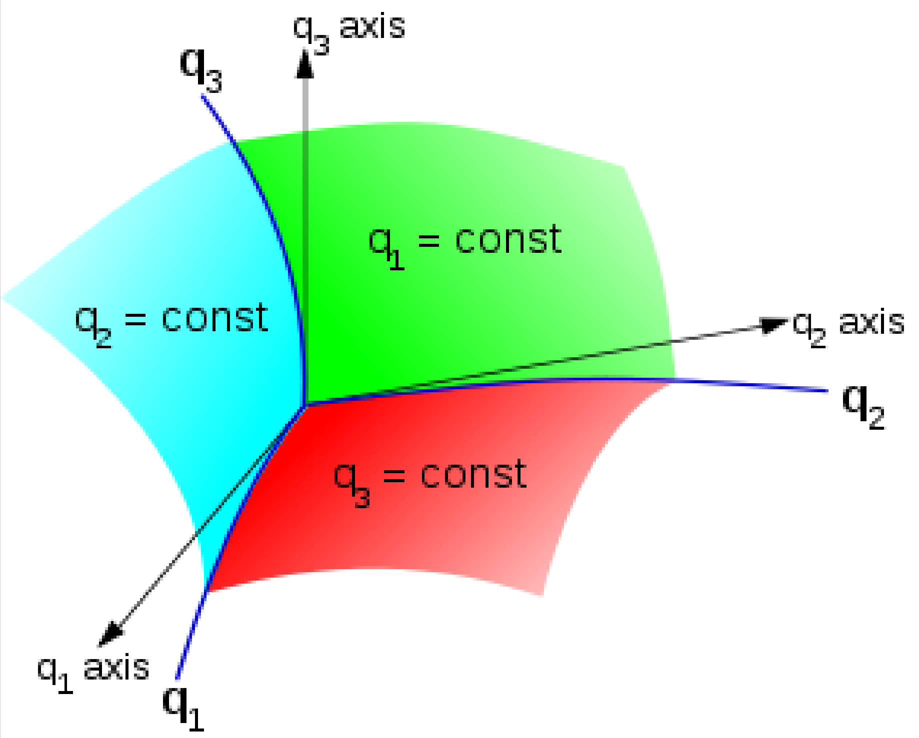


Help:
fingers should cross the positive part of the x-axis and the thumb should point in the direction of the z-axis

Coordinate systems – curvilinear coordinate system:

Beside the well known Cartesian coordinate system, there exist a variety of so called curvilinear (orthogonal) coordinate systems. There are known 11 curvilinear coordinate systems:

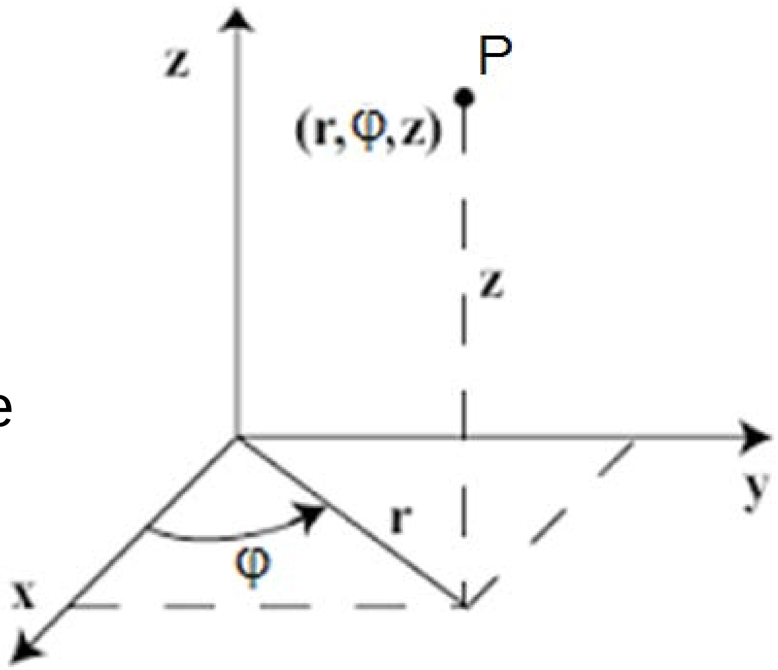
- cylindrical coordinates,
- spherical coordinates,
- elipsoidal coordinates,
- eliptic cylindrical coordinates,
- parabolic cylindrical coordinates,
- paraboloidal coordinates
- prolate spheroidal coordinates,
- oblate spheroidal coordinates,
- bipolar coordinates,
- toroidal coordinates,
- conical coordinates,



Coordinate systems – cylindrical coordinate system:

Coordinates: r, φ z

- the radial distance r is the Euclidean distance from the z axis to the point P.
- the azimuth φ is the angle between the reference direction on the chosen plane and the line from the origin to the projection of P on the plane.
- the height z is the signed distance from the chosen plane to the point P



transformation formulas (give the relation between the curvilinear and Cartesian system):

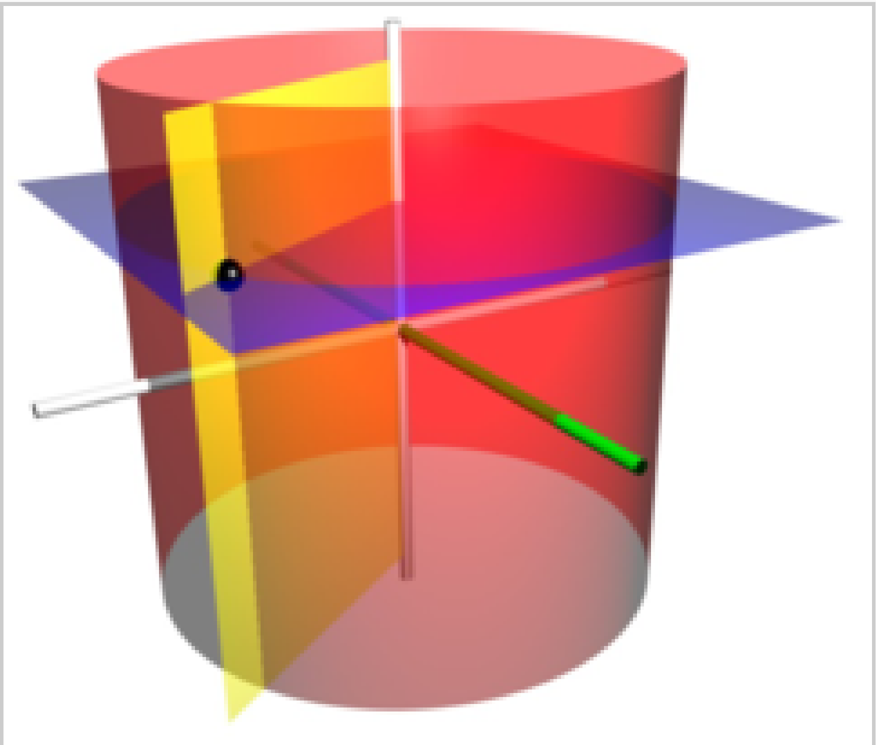
$$x = \cos\varphi, \quad y = r\sin\varphi, \quad z = z\,,$$

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \mathrm{arctg}(y/x), \quad z = z\,,$$

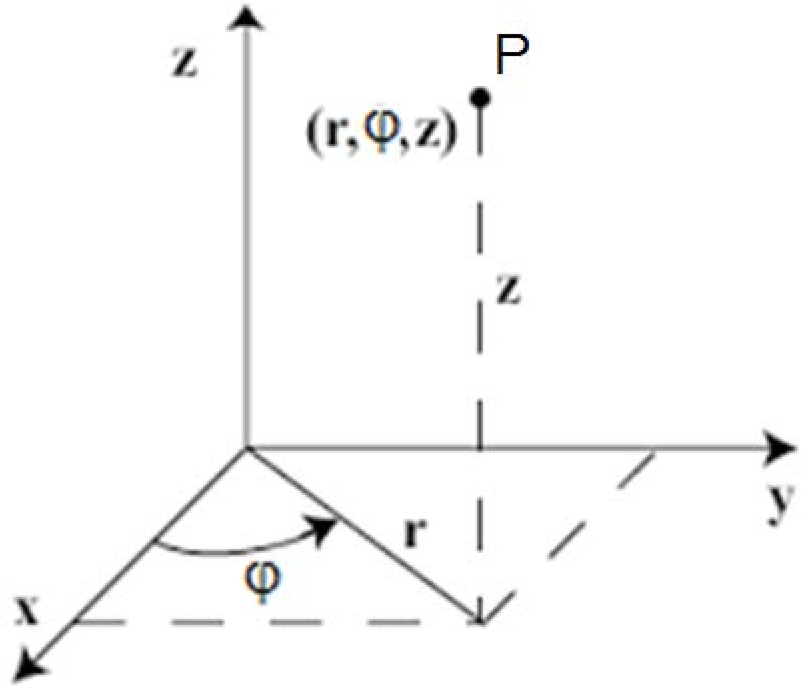
$$r \in \langle 0, +\infty), \quad \varphi \in \langle 0, 2\pi\rangle, \quad z \in (-\infty, +\infty)$$

Coordinate systems – cylindrical coordinate system:

so called coordinate surface (one coordinate is const.)



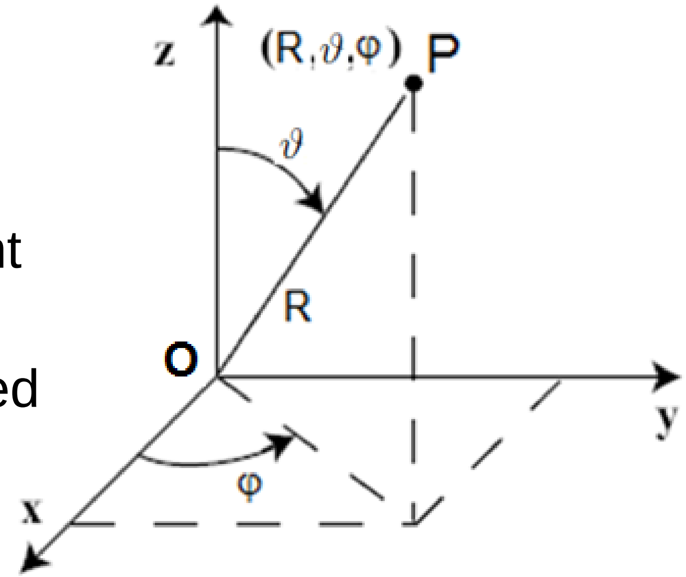
The coordinate surfaces of the cylindrical coordinates (r , φ, z). The red cylinder shows the points with r=2, the blue plane shows the points with z=1, and the yellow half-plane shows the points with φ=−60°.



Coordinate systems – spherical coordinate system:

Coordinates: R ϑ, φ

- the radius or radial distance R is the Euclidean distance from the origin O to P.
- the inclination (or polar angle) is the angle ϑ between the zenith direction and the line segment OP.
- the azimuth (or azimuthal angle) φ is the signed angle measured from the azimuth reference direction to the orthogonal projection of the line segment OP on the reference plane.



transformation formulas (give the relation between the curvilinear and Cartesian system):

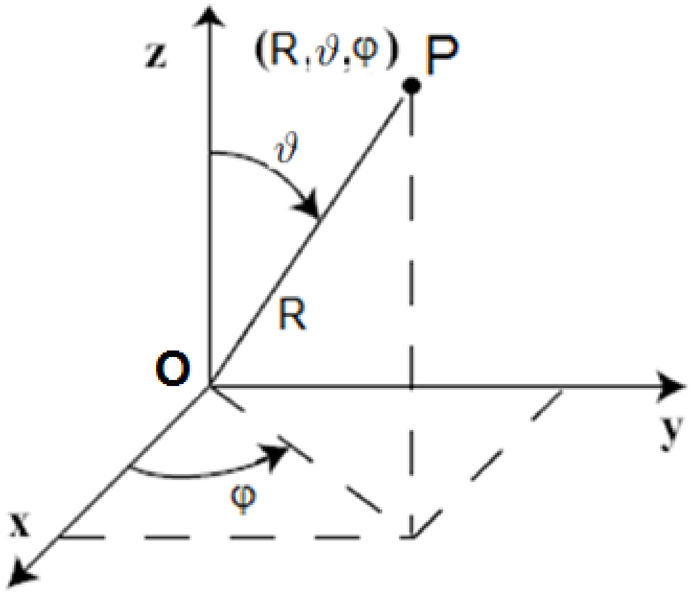
$$x = R\sin\vartheta\cos\varphi, \quad y = R\sin\vartheta\sin\varphi\,, \quad z = R\cos\vartheta$$

$$R = \left(x^2 + y^2 + z^2\right)^{1/2}, \quad \cos\vartheta = z/R\,, \quad \mathrm{tg}\,\varphi = y/x\,,$$

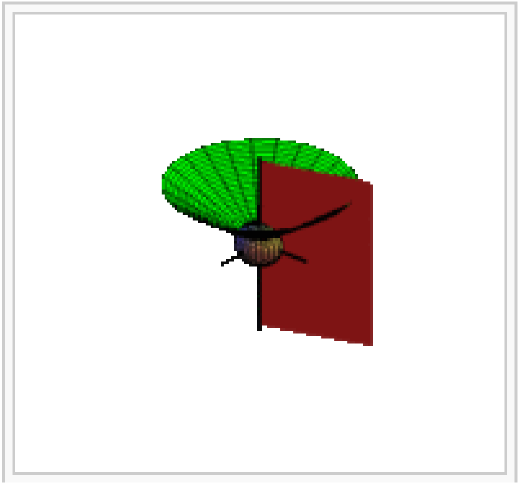
$$R \in \langle 0, +\infty)\,, \quad \vartheta \in \langle 0, \pi\rangle, \quad \varphi \in \langle 0, 2\pi\rangle\,.$$

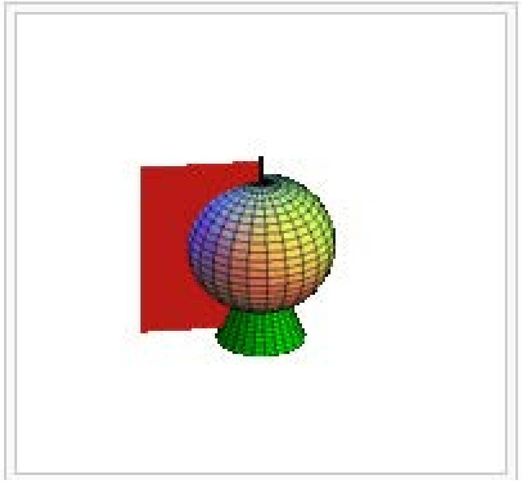
Coordinate systems – spherical coordinate system:

so called coordinate surface (one coordinate is const.)



nice animation:
https://en.wikipedia.org/wiki/Coordinate_system





two screen-shots from this animation

Lecture 2: Linear algebra, coordinate systems, vector analysis

Content:

- addition to previous lecture
- systems of linear equations, methods of solution
- coordinate systems
- vectors, basic definitions and operations

Vectors:

In mathematical physics for the description of physical quantities we recognise the following sequence of objects: **scalars**, **vectors** and **tensors**.

scalars (have only a magnitude or size)
(time, temperature, angle, length, ...) t

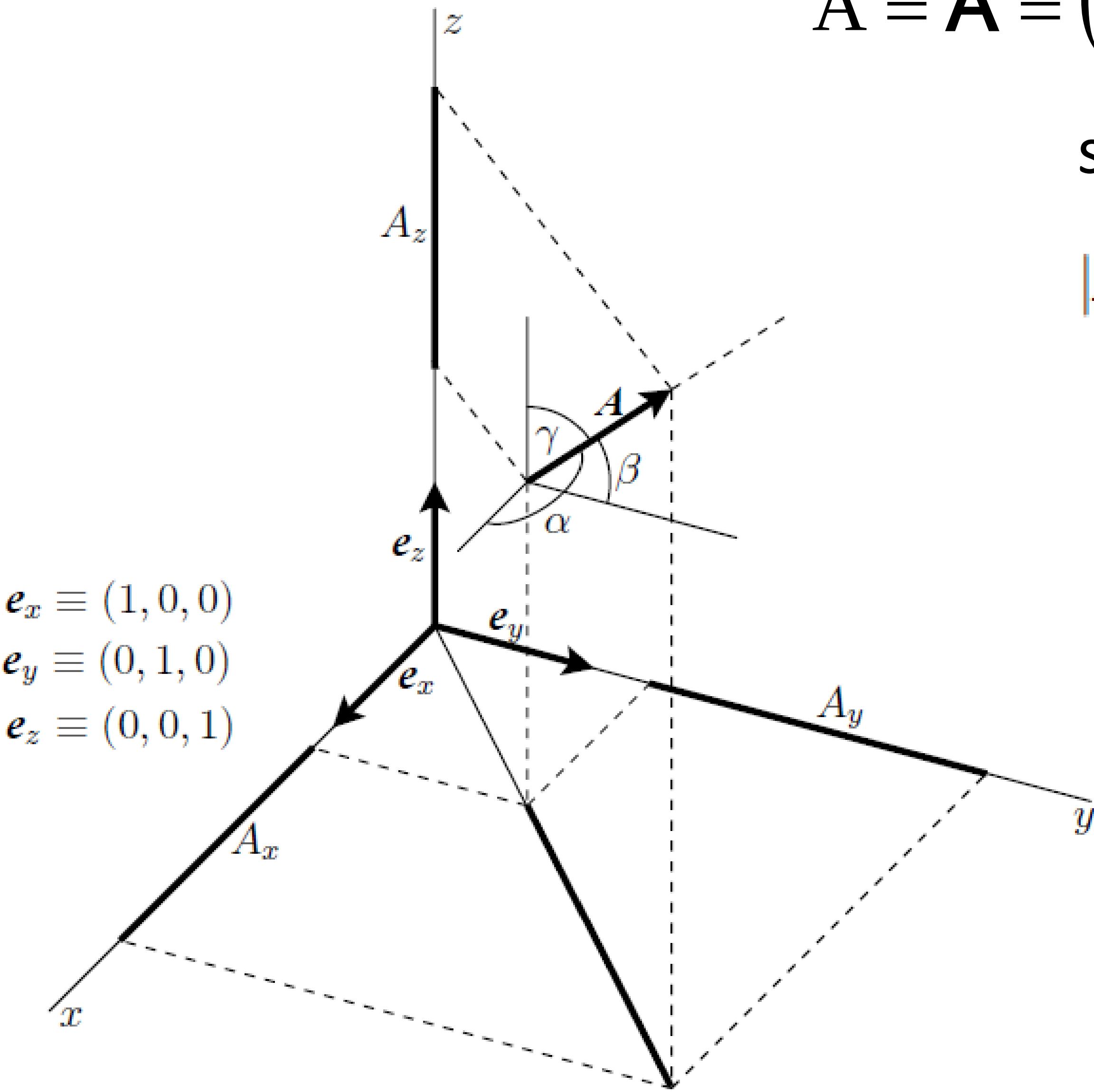
vectors (have magnitude and direction)
(strength, velocity, acceleration, ...) $\vec{F}$

tensors (generalisation of a vector – quantity has several dimensions) $\overline{T}$
(tensor of tension,...)

Vectors – basic properties and operations:

Vector is given by its components (in Carthesian coordinate system A_x, A_y, A_z):

$$\vec{A} = \mathbf{A} = \left(A_x, A_y, A_z\right)$$



size of the vector:

$$|A| = \left[A_x^2 + A_y^2 + A_z^2\right]^{1/2}$$

directional angles of the vector: α, β, γ

$$A_x = |A| \cos\alpha,$$

$$A_y = |A| \cos\beta,$$

$$A_z = |A| \cos\gamma.$$

Vectors – basic properties and operations:

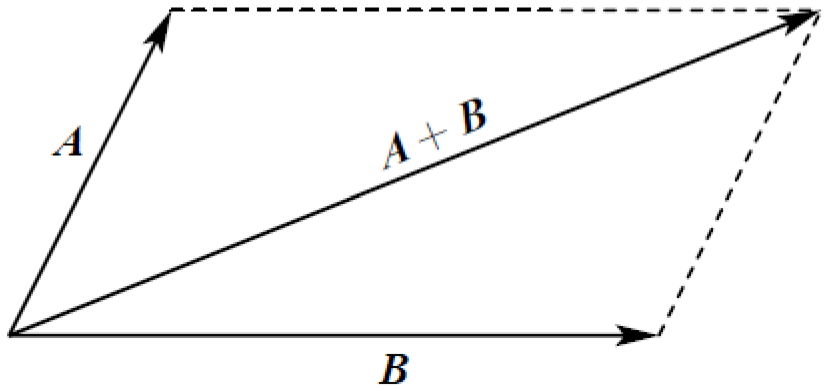
Multiplication of a vector **A** by a scalar f:

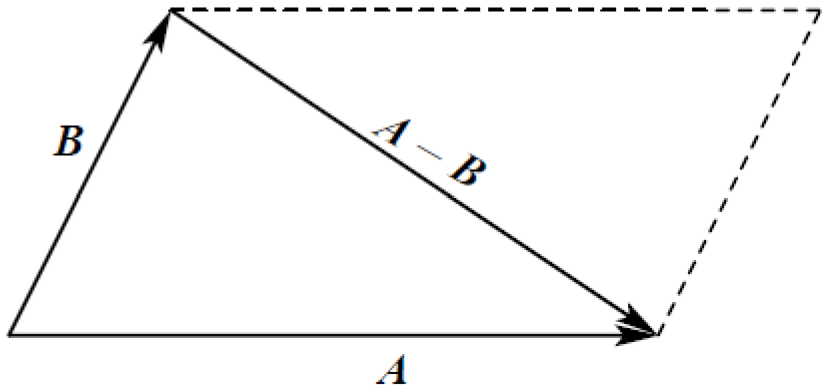
$$\boldsymbol{A} \cdot f = f \cdot \boldsymbol{A} = (f\, A_x \boldsymbol{i} + f\, A_y \boldsymbol{j} + f\, A_z \boldsymbol{k})$$

Addition and subtraction of two vectors (**A** and **B**):

$$\boldsymbol{A} \pm \boldsymbol{B} = (A_x \pm B_x)\,\boldsymbol{i} + (A_y \pm B_y)\,\boldsymbol{j} + (A_z \pm B_z)\,\boldsymbol{k}$$

graphical way...





Vectors – basic properties and operations:

Scalar multiplication of a vector **A** and **B**:

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$$

or (dot product)

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos \vartheta$$

where ϑ is the angle between these two vectors (**A** and **B**).

Result of this operation is a scalar (number).

When these vectors are orthogonal (angle between them is 90°), then scalar multiplication is equal to zero.

Comment: Scalar multiplication is commutative operation.

Vectors – basic properties and operations:

Vector multiplication of a vector **A** and **B**:

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{e}_x + (A_z B_x - A_x B_z)\boldsymbol{e}_y + (A_x B_y - A_y B_x)\boldsymbol{e}_z$$

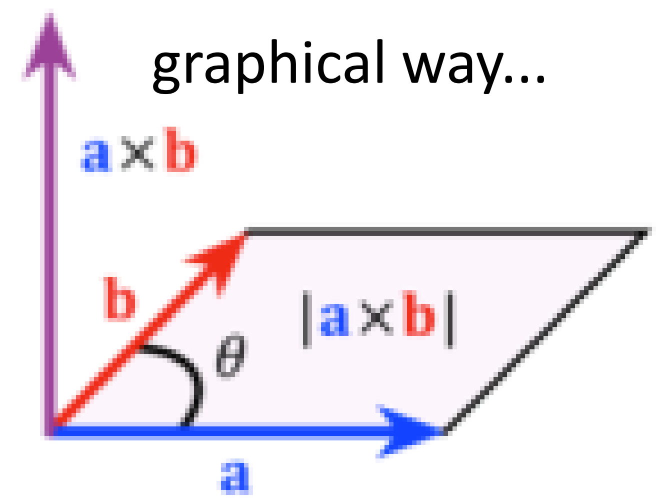
or (cross product)

$$|\boldsymbol{C}| = |\boldsymbol{A}|\,|\boldsymbol{B}| \sin\vartheta$$

where ϑ is the angle between these two vectors (**A** and **B**).

Result of this operation is a vector.

graphical way...



Comment: Vector multiplication is anti-commutative operation.

$$\boldsymbol{B} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{B}$$

Vectors – basic properties and operations:

Vector multiplication of a vector **A** and **B**:

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{e}_x + (A_z B_x - A_x B_z)\boldsymbol{e}_y + (A_x B_y - A_y B_x)\boldsymbol{e}_z$$

(cross product)

We can express this formula in a very compact form – as a determinant with 3 × 3 elements:

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vectors – basic properties and operations:

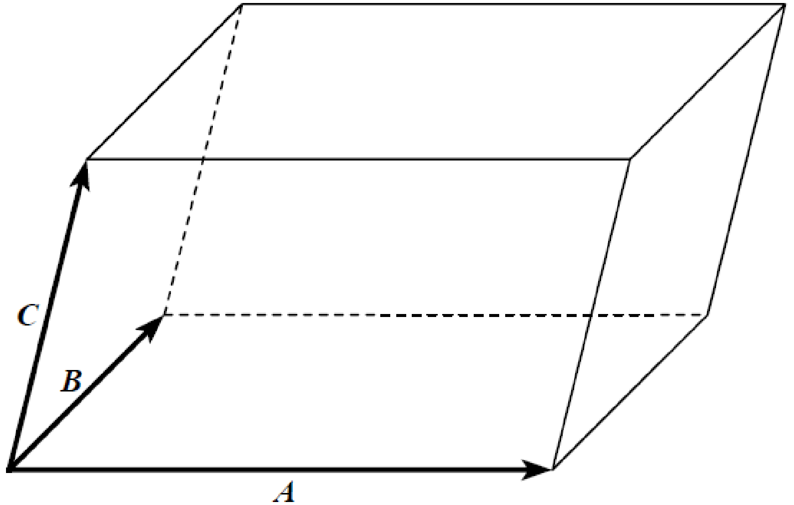
Mixed multiplication (triple product) of a vector **A** , **B** and **C**:

$$\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}.$$

$$\begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$

Result of this operation is a scalar (number).

It is the volume of a paralelipiped with a base given by **A** and **B** vectors and **C** is connected with its height.



Vectors – basic properties and operations:

Double vector multiplication of vectors **A** , **B** and **C**:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

Result of this operation is a vector.