Lecture 5: derivatives

Content:

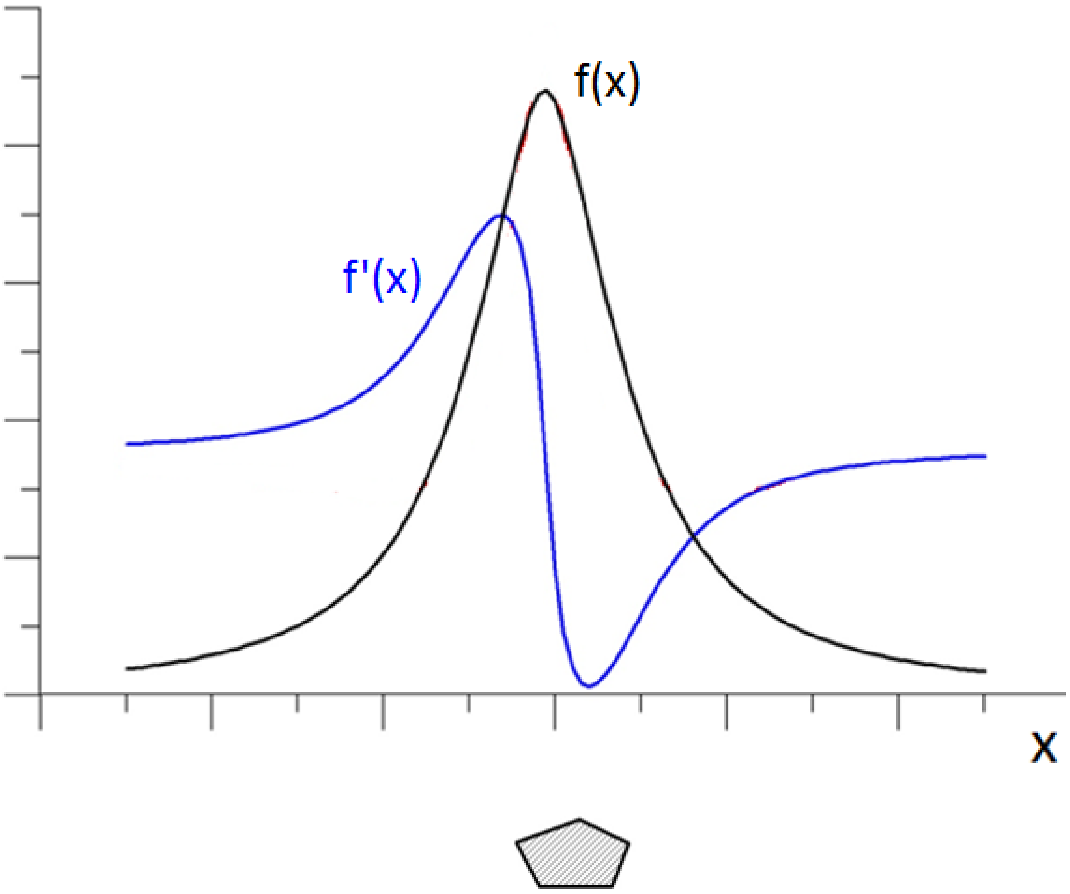
- derivatives – introduction
- derivatives, basic examples (using the definition)
- non-differentiable functions
- basic differentiation rules
- derivatives of elementary functions
- higher derivatives

Derivatives - introduction

- Together with integrals one of the most important tools in mathematical calculus.

- Derivative measures the sensitivity of the function f(*x*) to the change of the independent variable (*x*) - we can say that the derivative of a function f(*x*) of a variable *x* is a measure of the rate at which the value of the function changes with respect to the change of the variable. In a very simple form we can say that a derivative gives the change of function f(*x*) for a very small change of *x*. It is called the derivative of f with respect to *x*.

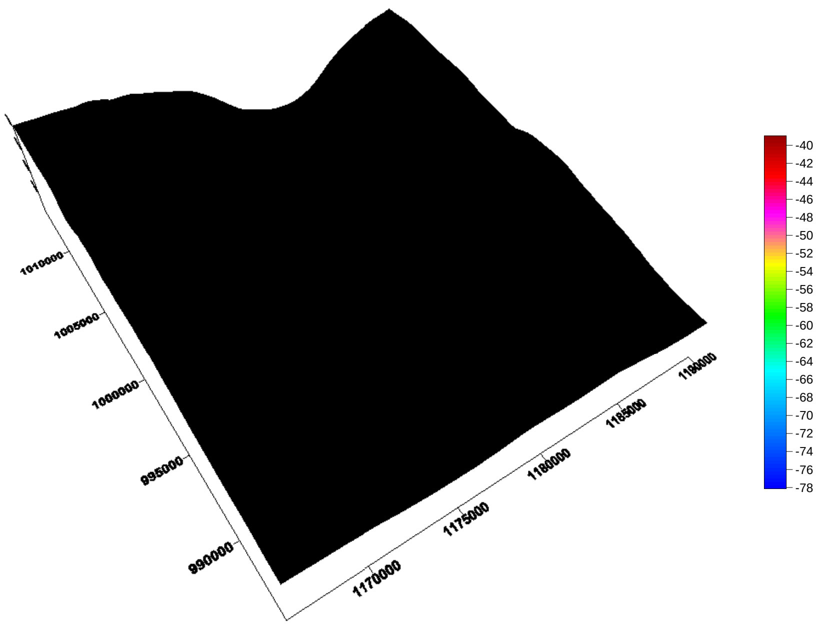
- Many applications in physics and other natural sciences are build upon the use of derivatives (e.g. derivative of the position of a moving object with respect to time is the object's velocity).



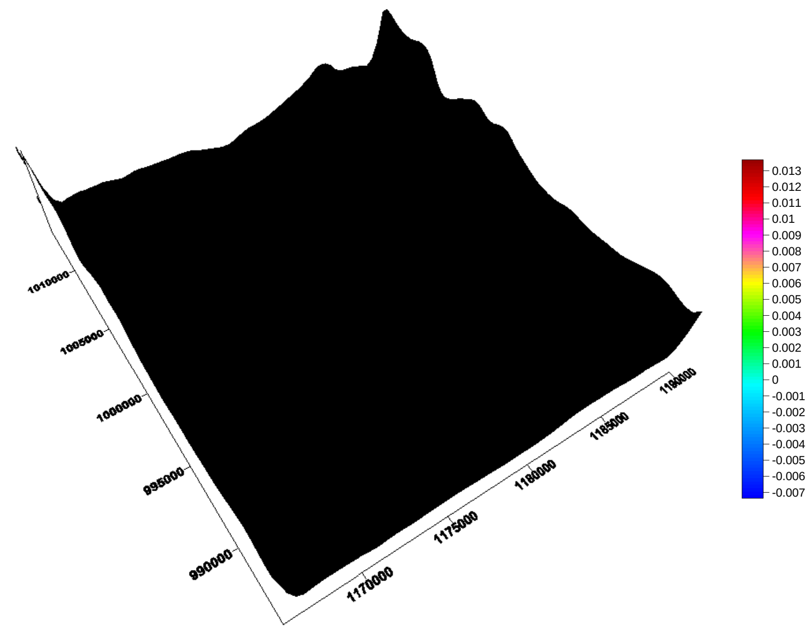
Example from physics:

Size of gravitational attraction of a mass f(x) (black curve) and its derivative f ’(x) (blue curve)

Comment: Both curves are plotted in one graph – this is incorrect (vertical scales are different).



anomalous gravity field (Dead Sea)



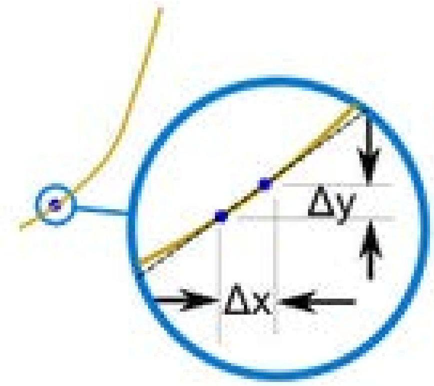
its derivative with respect to x

A next example from physics:

Map of anomalous gravitational attraction in the area of Dead Sea (left) and its derivative with respect to x-coordinate (right).

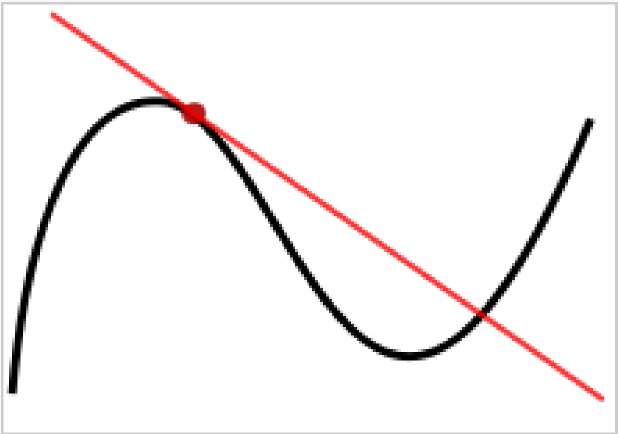
Derivatives - introduction

- derivative is often described as the "instantaneous rate of change“
- the process of finding a derivative is called differentiation. The reverse process is called antidifferentiation. The *fundamental theorem of calculus* states that antidifferentiation is the same as integration.



$$\text{slope } m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

- the derivative of a function of a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point; this means that it describes the best linear approximation of the function near that input value.



The graph of a function, drawn in black, and a tangent line to that function, drawn in red. The slope of the tangent line is equal to the derivative of the function at the marked point.

Derivatives - notation

Two distinct notations are commonly used for the derivative:

one coming from Gottfried Wilhelm Leibniz: $\frac{dy}{dx}$

and the second from Joseph Louis Lagrange: $y'(x)$

(sometimes signed also as Newton's notation - $\dot{y}$)

1. Leibniz's notation is suggesting the ratio of two infinitesimal quantities – this expression is read as: "the derivative of *y* with respect to *x*" or "*dy* by *dx*".
2. In Lagrange's notation the derivative with respect to *x* of a function *y*(*x*) is denoted *y*'(*x*) – this expression is read as: "*y* prime *x*" or "*y* dash *x*".

Comment: In mathematics, infinitesimals are things so small that there is no way to measure them.

Derivatives - notation

Sir Isaac Newton (left) and Gottfried Wilhelm von Leibniz (right)

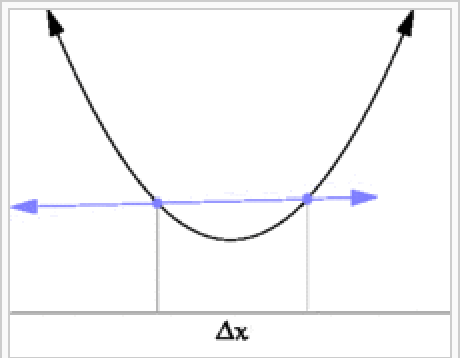
Leibniz–Newton calculus controversy

Derivatives – basic definition

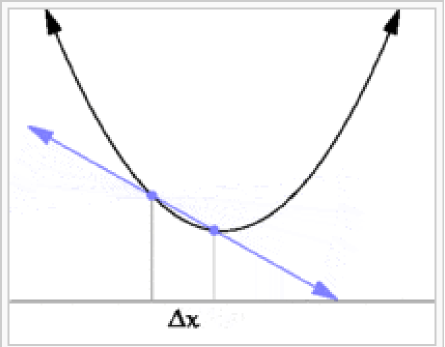
Definition:

The derivative of *f*(*x*) with respect to *x* is the function *f* ’(*x*):

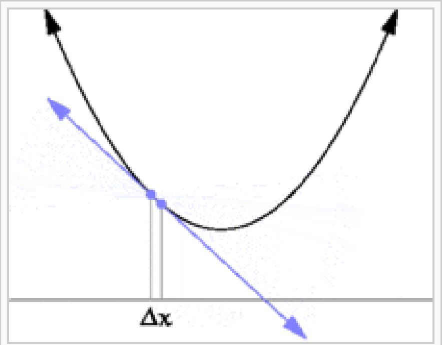
$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



A secant approaches a tangent



A secant approaches a tangent



A secant approaches a tangent

when Δx is getting smaller then the ratio of $\Delta f/\Delta x$ is getting closer to the slope (derivative) of the function in this point

link:

https://en.wikipedia.org/wiki/Derivative#Notation:

Derivatives – definition

Definition:

The derivative of *f*(*x*) with respect to *x* is the function *f* '(*x*):

$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

function f(x) is differentiable, when this limit exists and when it exists in every point of a defined interval.

Equivalent notations:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(x)_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

Comment:

Derivative of a function is again a function!

Example:

$(x^2)' = ?$

or

$d(x^2)/dx = ?$

Example: the function $\mathbf{f(x)} = \mathbf{x^2}$

We know $\mathbf{f(x)} = \mathbf{x^2}$, and can calculate $\mathbf{f(x+\Delta x)}$:

Start with: $\mathbf{f(x+\Delta x) = (x+\Delta x)^2}$

Expand $(x + \Delta x)^2$: $\mathbf{f(x+\Delta x) = x^2 + 2x\,\Delta x + (\Delta x)^2}$

The slope formula is: $$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Put in $\mathbf{f(x+\Delta x)}$ and $\mathbf{f(x)}$: $$\frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

Simplify (x^2 and $-x^2$ cancel): $$= \frac{2x\,\Delta x + (\Delta x)^2}{\Delta x}$$

Simplify more (divide through by Δx): $= \mathbf{2x + \Delta x}$

And then **as Δx heads towards 0** we get: $= 2x$

Result: the derivative of $\mathbf{x^2}$ is $\mathbf{2x}$

Derivatives – definition

- derivative is often described as the "instantaneous rate of change“

- in a very simple form we can say that a derivative gives the change of function f(*x*) for a very small change of *x*

- the derivative of a function of a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point

- and how is the relationship „differentiable vs. continuous“?

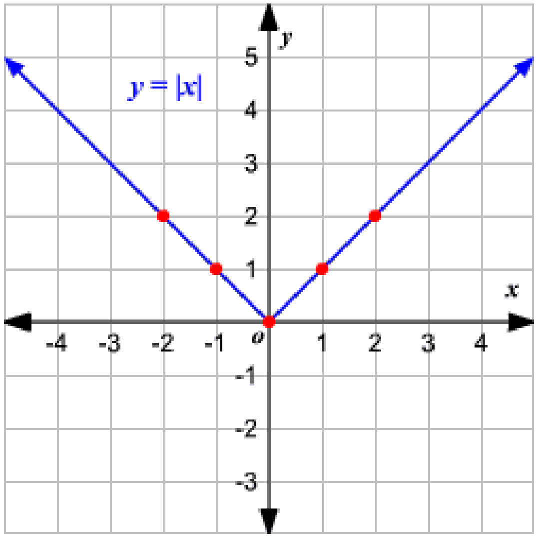
Differentiable implies Continuous

Theorem. *If a function f is differentiable at some a in its domain, then f is also continuous at a.*

But this statement is not a equivalency – if a function is continuous, it must not be also differentiable.

Derivatives – non-differentiable functions

Functions with „corner or sharp edge (cusp)“:



A graph with a corner. Consider the function

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

This function is continuous at all x, but it is not differentiable at $x = 0$.

To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \operatorname{sign}(x).$$

We know this limit does not exist.

If you look at the graph of $f(x) = |x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

Derivatives – non-differentiable functions

Functions with „corner or sharp edge (cusp)“:

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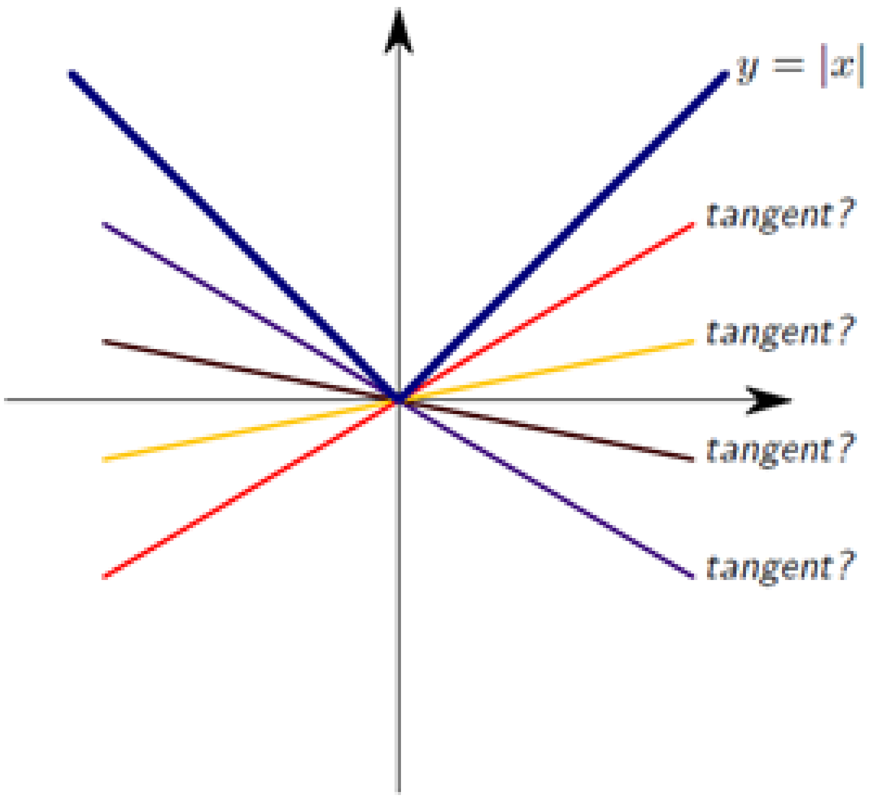

Figure The graph of $y = |x|$ has no tangent at the origin.

Derivatives – non-differentiable functions

Functions with „corner or sharp edge (cusp)“:

A graph with a cusp.

Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|x|}}{x} = ?$$

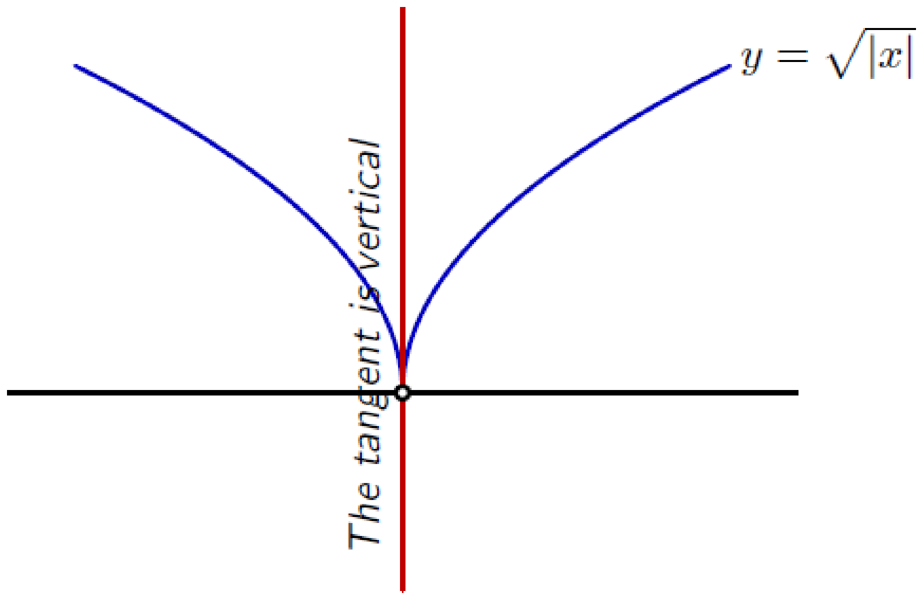

Figure Tangent to the graph of $y = |x|^{1/2}$ at the origin

Comment:

Somebody could argue the tangent is the vertical line (identical with the y-axis), but vertical lines do not have slopes (slope is infinity).

Derivatives – basic examples (using the definition)

2.1. Example – The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

Leibniz would have written

$$\frac{dx^2}{dx} = 2x.$$

2.2. The derivative of $g(x) = x$ is $g'(x) = 1$. Indeed, one has

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

In Leibniz' notation:

$$\frac{dx}{dx} = 1.$$

But be careful!

Results coming from Leibniz's rule can be misleading here!

Example (when Leibniz notation doesn‘t work):

Find the derivative of function $f(x) = x^3$.

Derivatives – basic examples (using the definition)

2.3. The derivative of any constant function is zero .

Let $k(x) = c$ be a constant function. Then we have

$$k'(x) = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

Leibniz would have said that if c is a constant, then

$$\frac{dc}{dx} = 0.$$

We have obtained following sequence of solutions:

$$(x^0)' = 0,$$

$$(x^1)' = 1,$$

$$(x^2)' = 2x,$$

$$(x^3)' = 3x^2,$$

... would it be possible to write here some kind of generalization?

Derivative of x^n, where $n = 1, 2, 3,...$ is equal to:

$$(x^n)' = nx^{n-1}\,.$$

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Derivatives – basic differentiation rules

As is was also in the case of limits evaluation, the use of the basic definition for the evaluation of derivatives is sometimes very cumbersome and time consuming, so we use some basic rules for differentiation.

1. differentiation of a constant (constant rule)
2. differentiation of constant multiple
3. sum or difference rule
4. product rule
5. quotient rule
6. chain rule
7. power rule

Derivatives – basic differentiation rules

As is was also in the case of limits evaluation, the use of the basic definition for the evaluation of derivatives is sometimes very cumbersome, so we use some basic rules for differentiation.

Constant rule: $c' = 0$ $\qquad \frac{dc}{dx} = 0$

Sum rule: $(u \pm v)' = u' \pm v'$ $\qquad \frac{du \pm v}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

Product rule: $(u \cdot v)' = u' \cdot v + u \cdot v'$ $\qquad \frac{duv}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

Quotient rule: $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$ $\qquad \frac{d\frac{u}{v}}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

later: power and chain rule

where *c* – constant, *u*(*x*) and *v*(*x*) are functions

Derivatives – basic differentiation rules

Proof of the product rule: $(u \cdot v)' = u' \cdot v + u \cdot v'$ (1/2)

From the basic formula for the derivative evaluation: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

We will express the numerator part of the limit (using $f(x) = u(x) \cdot v(x)$):

$$f(x) - f(a) = u(x)v(x) - u(a)v(a)$$

And reformulate it by adding and subtracting the expression $u(x) \cdot v(a)$:

$$= u(x)v(x) - \underline{u(x)v(a)} + \underline{u(x)v(a)} - u(a)v(a)$$
$$= u(x)\big(v(x) - v(a)\big) + \big(u(x) - u(a)\big)v(a)$$

Now divide by $x - a$ and let $x \to a$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} u(x)\frac{v(x) - v(a)}{x - a} + \frac{u(x) - u(a)}{x - a}v(a)$$

Derivatives – basic differentiation rules

Proof of the product rule: $(u \cdot v)' = u' \cdot v + u \cdot v'$ (2/2)

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} u(x)\frac{v(x)-v(a)}{x-a} + \frac{u(x)-u(a)}{x-a}v(a)$$

(use the limit properties)

$$= \left(\lim_{x\to a} u(x)\right)\left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a}\right) + \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a}\right)v(a)$$

$$= \boxed{u(a)v'(a) + u'(a)v(a),}$$

In this last step we have used that

$$\lim_{x\to a} \frac{u(x)-u(a)}{x-a} = u'(a) \quad \text{and} \quad \lim_{x\to a} \frac{v(x)-v(a)}{x-a} = v'(a)$$

and also that

$$\lim_{x\to a} u(x) = u(a)$$

Derivatives – basic differentiation rules

Derivation of the quotient rule: $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

The result of the division operation in the quotient we will sign with a new additional function *w*(*x*) = *u*(*x*)/*v*(*x*), from which follows: *u* = *w*·*v.*

By the product rule we have $w' \cdot v + w \cdot v' = u'$,

so that

$$w' = \frac{u' - w \cdot v'}{v} = \frac{u' - (u/v) \cdot v'}{v} = \boxed{\frac{u' \cdot v - u \cdot v'}{v^2}}.$$

Next important rule of differentiation:

Differentiating a constant multiple of a function . Note that the rule

$$(cu)' = cu'$$

follows from the Constant Rule and the Product Rule.

Example: $f(x) = 2x^4 - x^3 + 7 \qquad f'(x) = 8x^3 - 3x^2$

Derivatives – basic differentiation rules

Simple examples (product and quotient rule):

Differentiate the following function:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$x\left(3 + x^2\right)$$

Result: $3\left(1 + x^2\right)$

Differentiate the following function:

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$\frac{1 + x}{1 - x}$$

Result: $\frac{2}{(1 - x)^2}$

Derivatives – basic differentiation rules

Next important rule of differentiation:

Product rule with more than one factor.

If a function is given as the product of n functions, i.e.

$$f(x) = u_1(x) \times u_2(x) \times \cdots \times u_n(x),$$

then you can differentiate it by applying the product rule $n-1$ times (there are n factors, so there are $n-1$ multiplications.)

After the first step you would get

$$f' = u_1'(u_2 \cdots u_n) + u_1(u_2 \cdots u_n)'.$$

In the second step you apply the product rule to $(u_2 u_3 \cdots u_n)'$.

This yields

$$\begin{aligned} f' &= u_1' u_2 \cdots u_n + u_1 [u_2' u_3 \cdots u_n + u_2 (u_3 \cdots u_n)'] \\ &= u_1' u_2 \cdots u_n + u_1 u_2' u_3 \cdots u_n + u_1 u_2 (u_3 \cdots u_n)'. \end{aligned}$$

Continuing this way one finds after $n-1$ applications of the product rule that

$$(u_1 \cdots u_n)' = u_1' u_2 \cdots u_n + u_1 u_2' u_3 \cdots u_n + \cdots + u_1 u_2 u_3 \cdots u_n'.$$

Derivatives – basic differentiation rules

Next important rule of differentiation – the power rule:

The Power rule . If all n factors in the previous paragraph are the same, so that the function f is the n^{th} power of some other function,

$$f(x) = \big(u(x)\big)^n,$$

then all terms in the right hand side are the same, and, since there are n of them, one gets

$$\boxed{f'(x) = nu^{n-1}(x)u'(x),}$$

or, in Leibniz' notation,

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}.$$

Comment: This is a generalization of the formula for $(x^n)' = nx^{n-1}$, which we had in the slide nr. 15 of this lecture.

Very important is the differentiation of the internal function $u'(x)$ in the end of the final expression (we will come to it also later – it is coming from the chain rule).

The power rule is valid also for non-integer and negative n.

Derivatives – basic differentiation rules

Example (power rule):

$$f'(x) = nu^{n-1}(x)u'(x),$$

Derivative of the square root. The derivative of $f(x) = \sqrt{x} = x^{1/2}$ is

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

where we used the power rule with $n = 1/2$ and $u(x) = x$.

Next important rule of differentiation – following from the power rule:

The Power Rule for Negative Integer Exponents.

Suppose $n = -m$ where m is a positive integer. Then the Quotient Rule tells us that

$$(u^{-m})' = \left(\frac{1}{u^m}\right)' = -\frac{(u^m)'}{(u^m)^2}.$$

Derivatives – basic differentiation rules

Example (power rule for negative integer exponents):

$$\left(\frac{1}{u^m}\right)' = -\frac{(u^m)'}{(u^m)^2}.$$

Find the derivative of function:

$$f(x) = \frac{1}{x^4}$$

Solution:

$$f'(x) = -\frac{\left(x^4\right)'}{\left(x^4\right)^2} = -\frac{4x^3}{x^8} = -\frac{4}{x^5}$$

Alternative solution (using the standard power rule):

$$f'(x) = \left(x^{-4}\right)' = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

Derivatives – basic differentiation rules

Next <u>very</u> important rule of differentiation the chain rule: (1/2)

The Chain Rule

Composition of functions. Given two functions f and g, one can define a new function called the *composition of f and g*. The notation for the composition is $f \circ g$, and it is defined by the formula

$$f \circ g(x) = f\big(g(x)\big).$$

The domain of the composition is the set of all numbers x for which this formula gives you something well-defined.

$$z = f(y) = f\big(g(x)\big) = f \circ g(x).$$

One says that *the composition of f and g is the result of subsituting g in f*.

For instance, if $f(x) = x^2 + x$ and $g(x) = 2x + 1$ then

$$f \circ g(x) = f\big(2x+1\big) = (2x+1)^2 + (2x+1)$$

$$\text{and } g \circ f(x) = g\big(x^2 + x\big) = 2(x^2 + x) + 1$$

Note that $f \circ g$ and $g \circ f$ are not the same fucntion in this example (they hardly ever are the same).

Derivatives – basic differentiation rules

Next <u>very</u> important rule of differentiation the chain rule: (2/2)

Theorem (Chain Rule). *If f and g are differentiable, so is the composition $f \circ g$. The derivative of $f \circ g$ is given by*

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

The chain rule tells you how to find the derivative of the composition $f \circ g$ of two functions f and g provided you now how to differentiate the two functions f and g.

When written in Leibniz' notation the chain rule looks particularly easy. Suppose that $y = g(x)$ and $z = f(y)$, then $z = f \circ g(x)$, and the derivative of z with respect to x is the derivative of the function $f \circ g$. The derivative of z with respect to y is the derivative of the function f, and the derivative of y with respect to x is the derivative of the function g. In short,

$$\frac{dz}{dx} = (f \circ g)'(x), \quad \frac{dz}{dy} = f'(y) \quad \text{and} \quad \frac{dy}{dx} = g'(x)$$

so that the chain rule says

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}.$$

Derivatives – basic differentiation rules

Next example (power and chain rules):

$$f'(x) = nu^{n-1}(x)u'(x),$$

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

Find the derivative of function:

$$f(x) = \sqrt{3 + x^2} = \left(3 + x^2\right)^{1/2}$$

Solution:

$$f'(x) = \frac{1}{2}\left(3 + x^2\right)^{-1/2}\left(3 + x^2\right)' = \frac{1}{2}\left(3 + x^2\right)^{-1/2}(0 + 2x) = \frac{2x}{2\left(3 + x^2\right)^{1/2}} = \frac{x}{\sqrt{3 + x^2}}$$

Next example (power and chain rules):

$$f'(x) = nu^{n-1}(x)u'(x),$$

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

Find the derivative of function:

$$f(x) = \frac{x-a}{\sqrt{x^2+b}}$$

Solution:

$$\left[\frac{x-a}{\sqrt{x^2+b}}\right]' = \frac{(x-a)'\sqrt{x^2+b} - (x-a)\left(\sqrt{x^2+b}\right)'}{\left(\sqrt{x^2+b}\right)^2} =$$

$$= \frac{1 \cdot \sqrt{x^2+b} - (x-a)\frac{1}{2}\left(x^2+b\right)^{-1/2} 2x}{\left(x^2+b\right)} =$$

$$= \frac{\sqrt{x^2+b} - x(x-a)\left(x^2+b\right)^{-1/2}}{x^2+b} =$$

Derivative Table

1. $$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

2. $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

3. $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

4. $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}$$

5. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

6. **(Chain rule)** If y = f(u) is differentiable on u = g(x) and u = g(x) is differentiable on point x, then the composite function y = f(g(x)) is differentiable and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

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Derivatives – elementary functions

Trigonometric functions: *sin*(*x*) 1/2

$$(\sin x)' = \cos x \quad \text{or} \quad \frac{d \sin x}{dx} = \cos x$$

PROOF. By definition one has

$$\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

To simplify the numerator we use the trigonometric addition formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

with $\alpha = x$ and $\beta = h$, which results in

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h) - 1}{h}$$

Derivatives – elementary functions

Trigonometric functions: *sin*(*x*) 2/2

Hence by the formulas

$$\lim_{h\to 0}\frac{\sin(h)}{h}=1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h}=0$$

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}+\sin(x)\frac{\cos(h)-1}{h}\\ &= \cos(x)\cdot 1+\sin(x)\cdot 0\\ &= \cos(x).\end{aligned}$$

A similar computation leads to the derivative of $\cos x$.

$$(\cos x)' = -\sin x \quad \text{or} \quad \frac{d\cos x}{dx} = -\sin x$$

Derivatives – elementary functions

Trigonometric functions: *tan*(*x*)

To find the derivative of $\tan x$ we apply the quotient rule to

$$\tan x = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)}.$$

We get

$$\tan'(x) = \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \boxed{\frac{1}{\cos^2(x)}}$$

Result can be expressed by means of the secant function (*sec x* = 1/*cos x*):

$$\frac{1}{\cos^2 x} = \sec^2 x$$

Trigonometric functions: co*t*(*x*)

$$(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Similarly, the result is expressed by the cosecant function (*csc x* = 1/*sin x*):

Derivatives – elementary functions

Next functions differentiations will be given in a form of a table (without derivations):

$y = f(x)$	$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$

Next examples (trigonometric function + power/chain rule):

Find the derivative of function:

$$f(x) = \sin^3 x$$

Solution:

$$\left(\sin^3 x\right)' = 3\sin^2 x\cos x$$

Find the derivative of function:

$$f(x) = \cos\left(3x^2 + 2x + 1\right)$$

Solution:

$$\left[\cos\left(3x^2 + 2x + 1\right)\right]' = -(6x + 2)\sin\left(3x^2 + 2x + 1\right)$$

Derivatives – elementary functions

Next functions differentiations will be given in a form of a table (without derivations):

$$\frac{d}{dx} a^x = (\ln a)\, a^x$$

(If $a = e$) $$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \log_a x = \frac{1}{(\ln a)\, x}$$

(If $a = e$) $$\frac{d}{dx} \ln x = \frac{1}{x}$$

Comment: Derivations of these expressions are little bit more complicated and will be shown later on.

Derivatives of elementary functions

$$\frac{dc}{dx} = 0$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}x^x = x^x(1 + \ln x)$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}\cot x = -\csc^2 x = -\frac{1}{\sin^2 x}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1 + x^2}$$

Find Derivative of $y = x^x$

A calculus tutorial on how to find the first derivative of $y = x^x$ for $x > 0$.

Note that the function defined by $y = x^x$ is neither a power function of the form x^k nor an exponential function of the form b^x and the formulas of Differentiation of these functions cannot be used. We need to find another method to find the first derivative of the above function.

If $y = x^x$ and $x > 0$ then $\ln y = \ln (x^x)$

Use properties of logarithmic functions to expand the right side of the above equation as follows.

$\ln y = x \ln x$

We now differentiate both sides with respect to x, using chain rule on the left side and the product rule on the right.

$y '(1 / y) = \ln x + x(1 / x) = \ln x + 1$, where $y ' = dy/dx$

Multiply both sides by y

$y ' = (\ln x + 1)y$

Substitute y by x^x to obtain

$y ' = (\ln x + 1)x^x$

http://www.analyzemath.com/calculus/Differentiation/first_derivative.html

Lecture 5: derivatives

Content:

- derivatives – introduction
- derivatives, basic examples (using the definition)
- non-differentiable functions
- basic differentiation rules
- derivatives of elementary functions
- higher derivatives

Higher derivatives

Derivative of a function is again a function – so it can be differentiated again (and again).

So, we can obtain a second derivative of *f*(*x*), third ..., *n*th.

In Lagrange's notation:.

$$f^{(0)} = f, \qquad f^{(1)} = f', \qquad f^{(2)} = f'', \qquad f^{(3)} = f''', \ldots . \qquad f^{(n)}.$$

In Leibniz's notation:.

$$\frac{d^n y}{dx^n} = f^{(n)}(x).$$

Example. If $f(x) = x^2 - 2x + 3$ then

$$f(x) = x^2 - 2x + 3$$
$$f'(x) = 2x - 2$$
$$f''(x) = 2$$
$$f^{(3)}(x) = 0$$
$$f^{(4)}(x) = 0$$