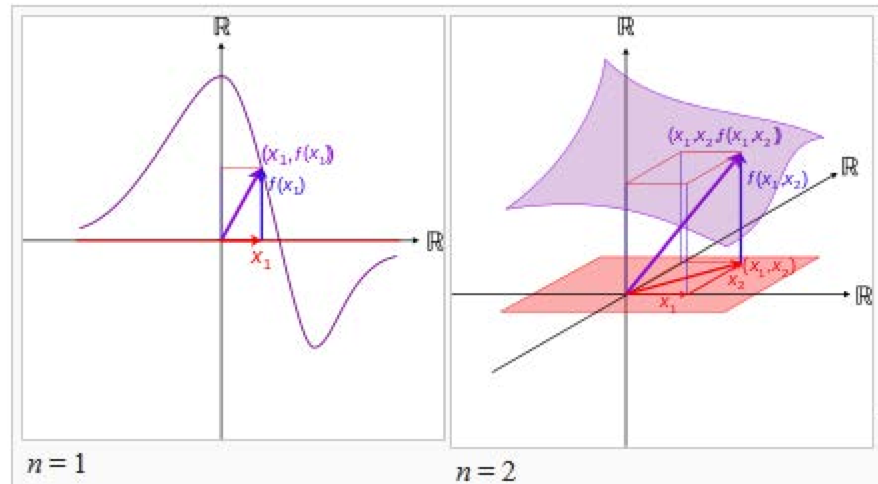
Lecture 9: functions of several variables

Content:

- basic definitions and properties
- partial and total differentiation
- differential operators
- multiple integrals
- examples of multiple integrals

Functions of several variables:

Previously we have studied functions of one variable, y = f(x) in which x was the independent variable and y was the dependent variable. We are going to expand the idea of functions to include functions with more than one independent variable. For example, consider the functions below:



$$f(x,y) = 2x^2 + y^2$$

or

$$g(x,y,z) = 2xe^{yz}$$

or

$$h(x_1,x_2,x_3,x_4) = 2x_1 - x_2 + 4x_3 + x_4$$

In more rigorous mathematical language:

$$z : \mathbb{R}^2 \to \mathbb{R}$$
$$z(x,y) = ax + by$$

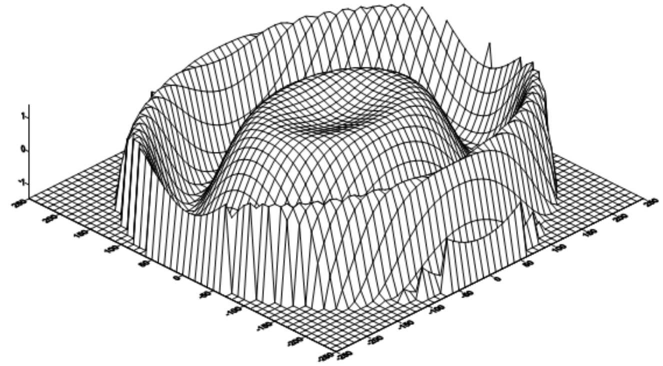
where a and b are real non-zero constants

$$z : \mathbb{R}^p \to \mathbb{R}$$
$$z(x_1, x_2, \ldots, x_p) = a_1x_1 + a_2x_2 + \cdots + a_px_p$$

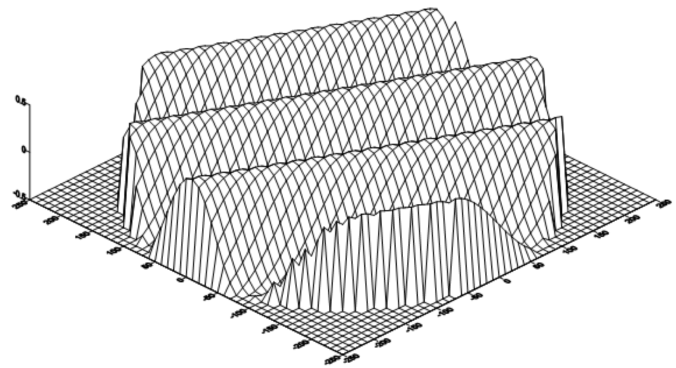
for p non-zero real constants $a_1, a_2, \ldots, a_p$

Functions of several variables:

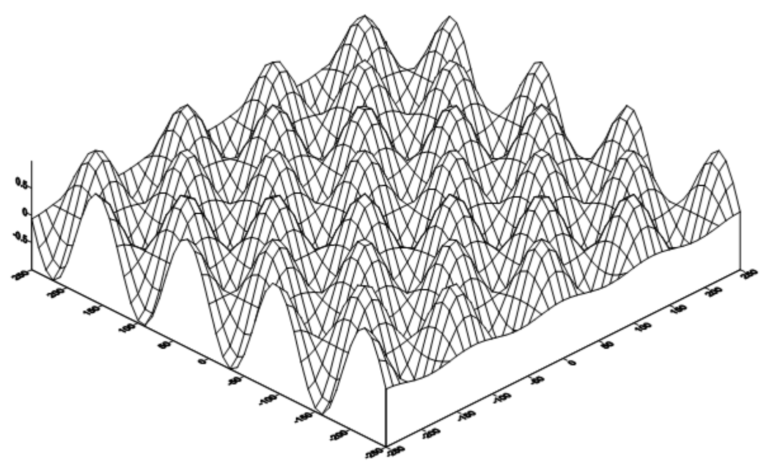
examples of graphs for $f = f(x,y)$



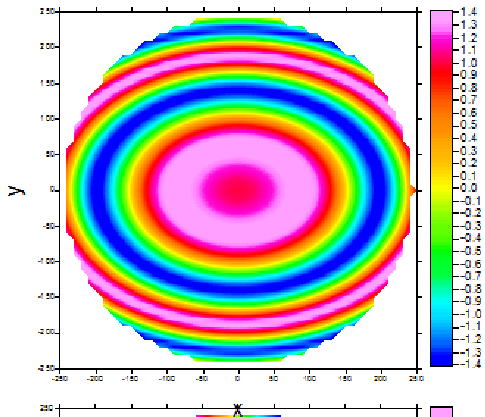
sin(0.0001*x^2+0.0002*y^2)+cos(0.0001*x^2+0.0002*y^2)

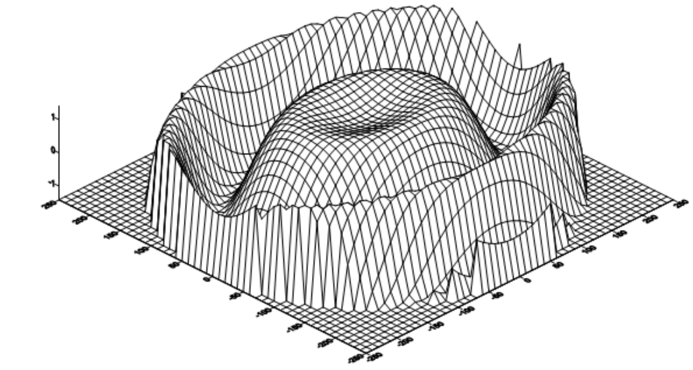


z = sin(0.01*x+0.02*y)*cos(0.01*x+0.02*y)

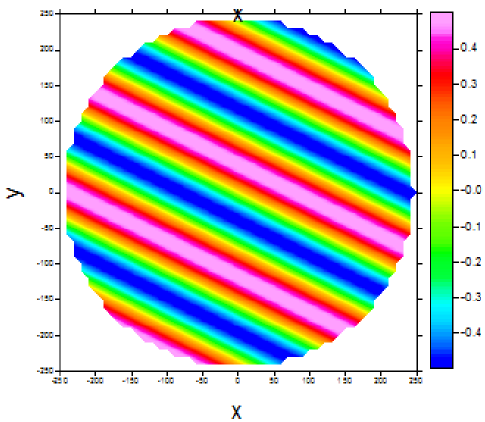


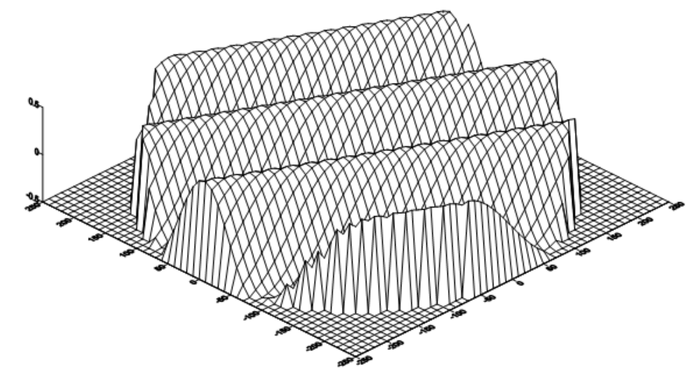
z = cos(0.05*x)*sin(0.05*y)



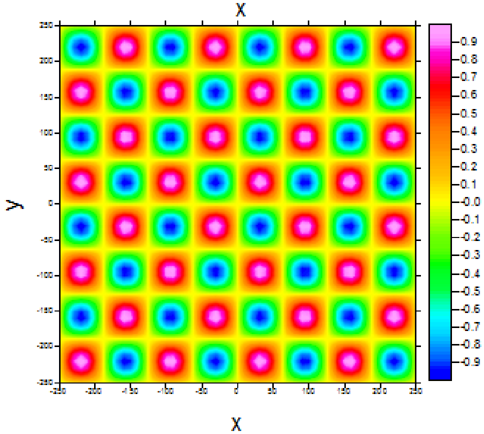


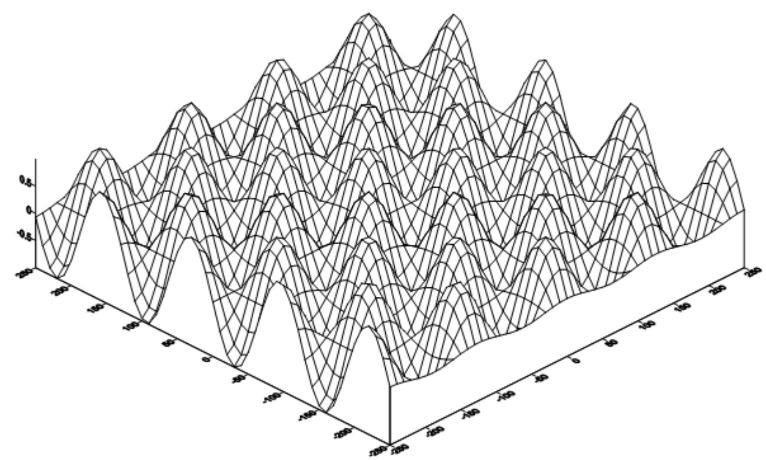
$\sin(0.0001*x^2+0.0002*y^2)+\cos(0.0001*x^2+0.0002*y^2)$





$z = \sin(0.01*x+0.02*y)*\cos(0.01*x+0.02*y)$



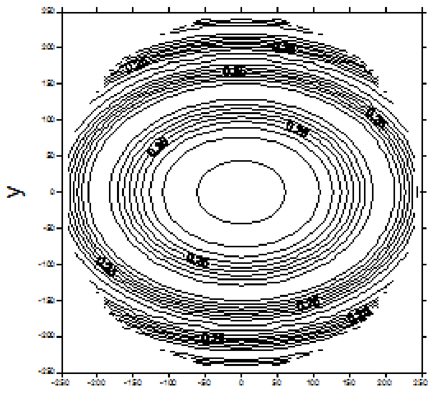


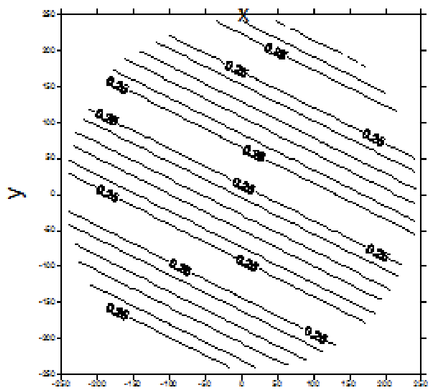
$z = \cos(0.05*x)*\sin(0.05*y)$

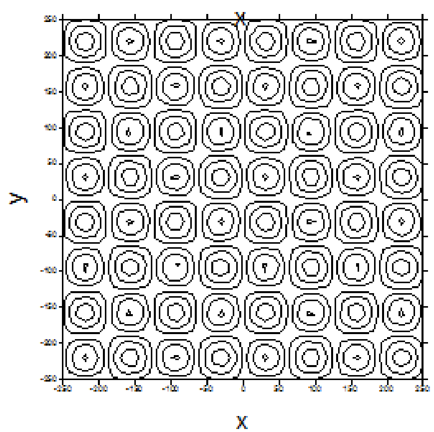
another kind of visualization

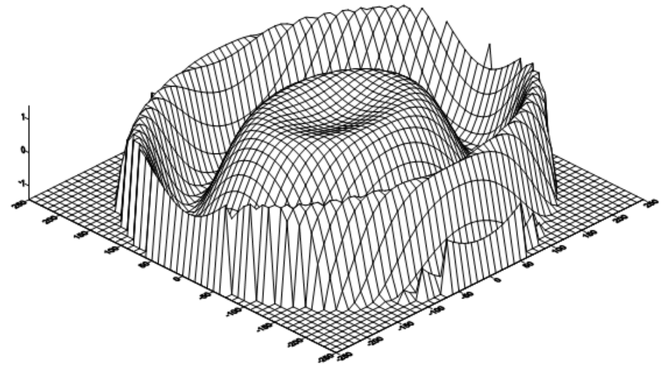
- so called <u>coloured image maps</u>

(there exist also so called contour maps)

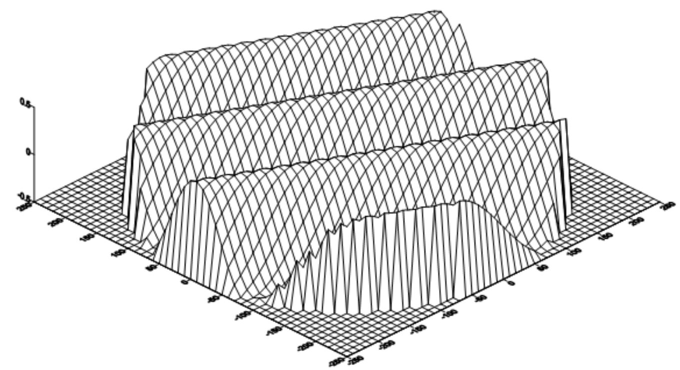




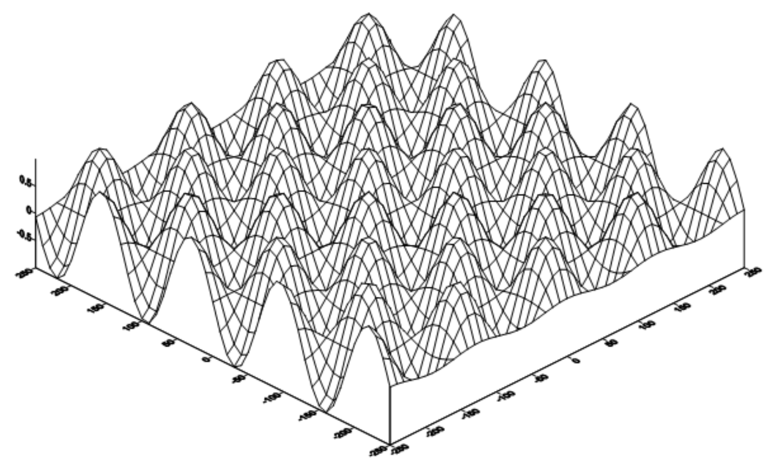




sin(0.0001*x^2+0.0002*y^2)+cos(0.0001*x^2+0.0002*y^2)

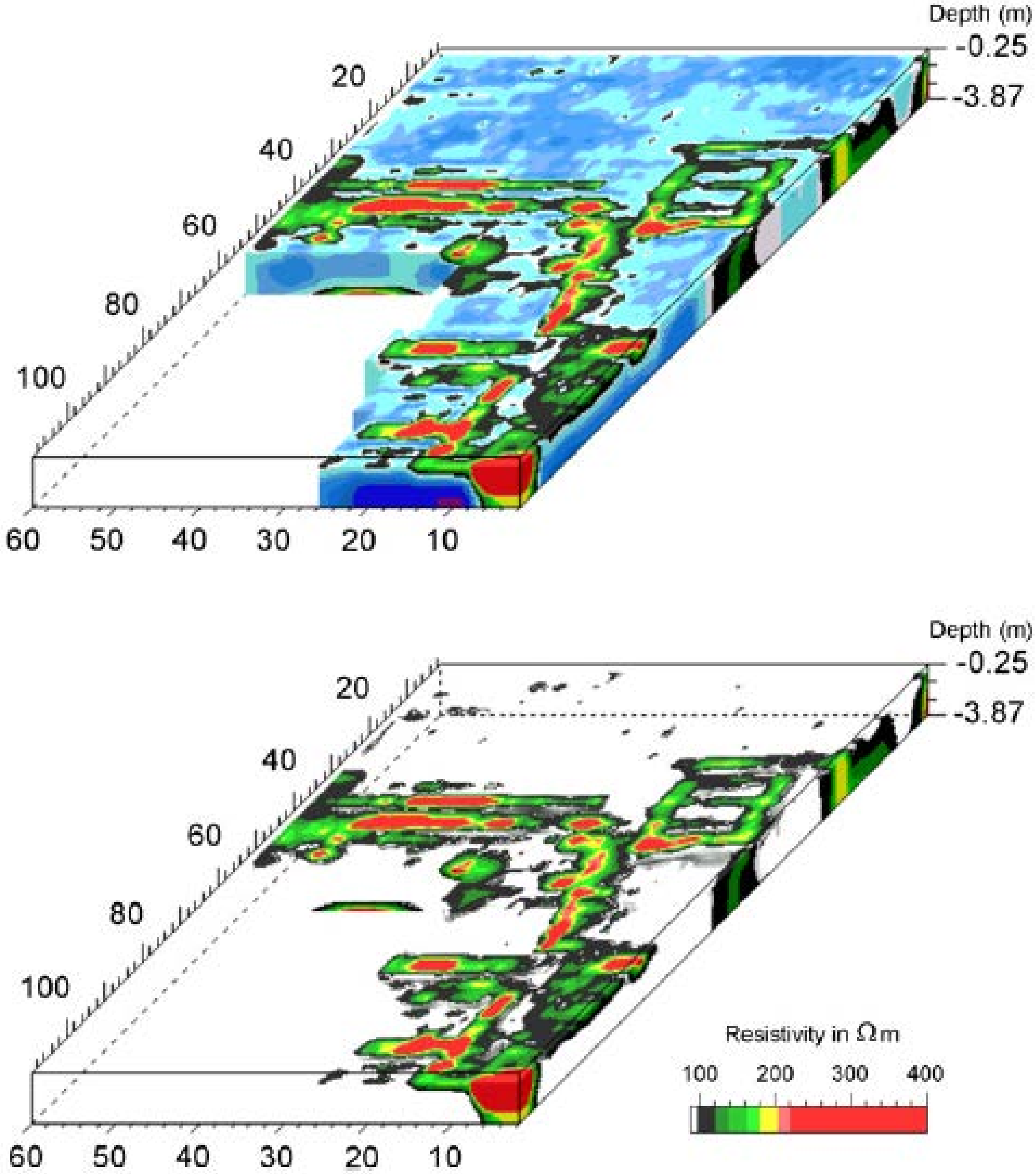


z = sin(0.01*x+0.02*y)*cos(0.01*x+0.02*y)



z = cos(0.05*x)*sin(0.05*y)

another kind of visualization
- so called coloured image maps
(there exist also so called <u>contour maps</u>)



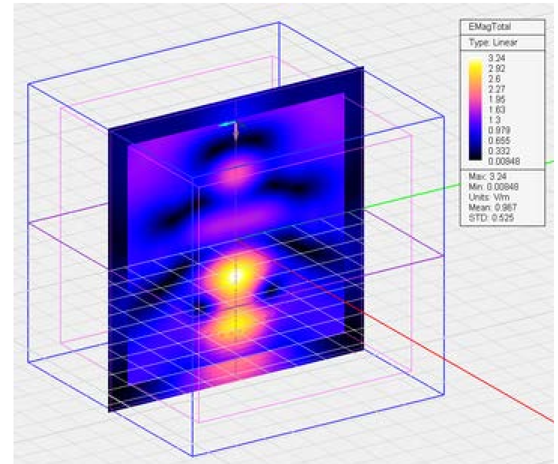
functions $f = f(x,y,z)$ are often visualized in form of voxel maps

Functions of several variables:

Functions of several variables are used in science for the description of various fields (physical fields, fields of properties ...).

scalar fields:

e.g. temperature, density,
concentration, electric charge, ...
$t(x,y,z)$, $\rho(x,y,z)$, $U(x,y,z)$,...



and also vector fields:

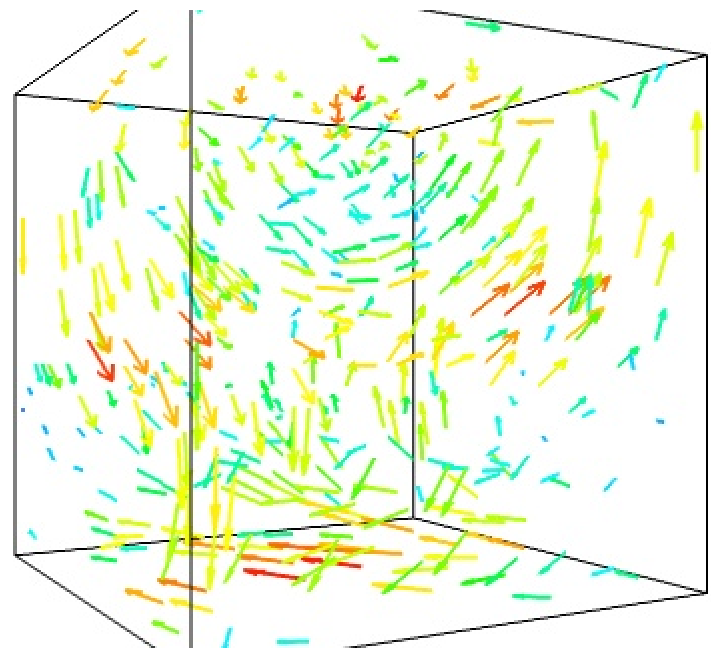
e.g. electrical intensity, fluid velocity,
gravitational acceleration,...

$$\vec{A} = [A_x, A_y, A_z]$$

$$A_x = A_x(x,y,z)$$

$$A_y = A_y(x,y,z)$$

$$A_z = A_z(x,y,z)$$



Functions of several variables:

Many properties are identical with the case of a function with one variable.

Limits and Continuity

- We say that a function $f(x, y)$ has limit L as (x, y) approaches a point (a, b) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

  if we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).

- We write also $f(x, y) \to L$ as $(x, y) \to (a, b)$ and

$$\lim_{x\to a, y\to b} f(x, y) = L$$

Functions of several variables:

Many properties are identical with the case of a function with one variable.

Continuity

- A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

- **Examples:** polynomials, rational, trigonometric, exponential, logarithmic functions are continuous on theirs domain.

With the continuity is connected also the so called distance function *d*:

$$d(\boldsymbol{x}, \boldsymbol{y}) = d(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Functions of several variables:

Some properties are new (compared with a function with one variable).

Symmetry:

A symmetric function is a function f is unchanged when two variables x_i and x_j are interchanged:

$$f(\ldots, x_i, \ldots, x_j, \ldots) = f(\ldots, x_j, \ldots, x_i, \ldots)$$

where *i* and *j* are each one of 1, 2, ..., *n*.

For example:

$$f(x, y, z, t) = t^2 - x^2 - y^2 - z^2$$

is symmetric in *x*, *y*, *z* since interchanging any pair of *x*, *y*, *z* leaves *f* unchanged, but is not symmetric in all of *x*, *y*, *z*, *t*, since interchanging *t* with *x* or *y* or *z* is a different function.

Lecture 9: functions of several variables

Content:

- basic definitions and properties
- partial and total differentiation
- differential operators
- multiple integrals
- examples of multiple integrals

Functions of several variables:

Some properties are new (compared with a function with one variable).

Partial derivatives:

In the case of functions of several variables, we recognize:

a) total derivative (all variables can vary and derivatives with respect to all variables are involved)

b) partial derivative (it is a derivative with respect to one of the variables with the others held constant)

$$f'_x,\ f_x,\ \partial_x f,\ \frac{\partial}{\partial x}f,\ \text{or}\ \frac{\partial f}{\partial x}$$

In the notation, Leibniz rule is used more often
(symbol ∂ is derived from "d" and it was introduced by Legendre – it is called as partial derivative symbol).

Example, function $f = x^2+xy+y^2$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2+xy+y^2\right) = 2x+y+0 = 2x+y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2+xy+y^2\right) = 0+x+2y = x+2y$$

another tool is given in the next slide:

Partial derivatives:

For the beginner it is helpful to imagine instead of a variable (e.g. *y*) for a moment a constant (e.g. *b*).

Example 1

Let $f(x,y) = y^3x^2$. Calculate $\frac{\partial f}{\partial x}(x,y)$.

Solution: To calculate $\frac{\partial f}{\partial x}(x,y)$, we simply view y as being a fixed number and calculate the ordinary derivative with respect to x. The first time you do this, it might be easiest to set $y = b$, where b is a constant, to remind you that you should treat y as though it were number rather than a variable. Then, the partial derivative $\frac{\partial f}{\partial x}(x,y)$ is the same as the ordinary derivative of the function $g(x) = b^3x^2$. Using the rules for ordinary differentiation, we know that

$$\frac{dg}{dx}(x) = 2b^3x.$$

Now, we remember that $b = y$ and substitute y back in to conclude that

$$\frac{\partial f}{\partial x}(x,y) = 2y^3x.$$

Partial derivatives – few examples:

1. If $z = f(x, y) = x^4y^3 + 8x^2y + y^4 + 5x$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = 4x^3y^3 + 16xy + 5 \qquad \text{(Note: } y \text{ fixed, } x \text{ independent variable, } z \text{ dependent variable)}$$

$$\frac{\partial z}{\partial y} = 3x^4y^2 + 8x^2 + 4y^3 \qquad \text{(Note: } x \text{ fixed, } y \text{ independent variable, } z \text{ dependent variable)}$$

2. If $z = f(x, y) = (x^2 + y^3)^{10} + \ln(x)$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = 20x(x^2 + y^3)^9 + \frac{1}{x} \qquad \text{(Note: We used the chain rule on the first term)}$$

$$\frac{\partial z}{\partial y} = 30y^2(x^2 + y^3)^9 \qquad \text{(Note: Chain rule again, and second term has no } y\text{)}$$

3. If $z = f(x, y) = xe^{xy}$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = e^{xy} + xye^{xy} \qquad \text{(Note: Product rule (and chain rule in the second term)}$$

$$\frac{\partial z}{\partial y} = x^2e^{xy} \qquad \text{(Note: No product rule, but we did need the chain rule)}$$

Functions of several variables:

Some properties are new (compared with a function with one variable).

Total derivative (differential):

In the case of functions of several variables, we recognize:

a) total derivative (all variables can vary and derivatives with respect to all variables are involved)

b) partial derivative (it is a derivative with respect to one of the variables with the others held constant)

For a function z = f(x, y, .. , u) the total differential is defined as

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \cdots + \frac{\partial z}{\partial u} du \,.$$

Example, function $f = x^2+xy+y^2$:

$$df = \frac{\partial}{\partial x}\left(x^2 + xy + y^2\right)dx + \frac{\partial}{\partial y}\left(x^2 + xy + y^2\right)dy = (2x + y)dx + (2y + x)dy$$

Differential operators:

There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

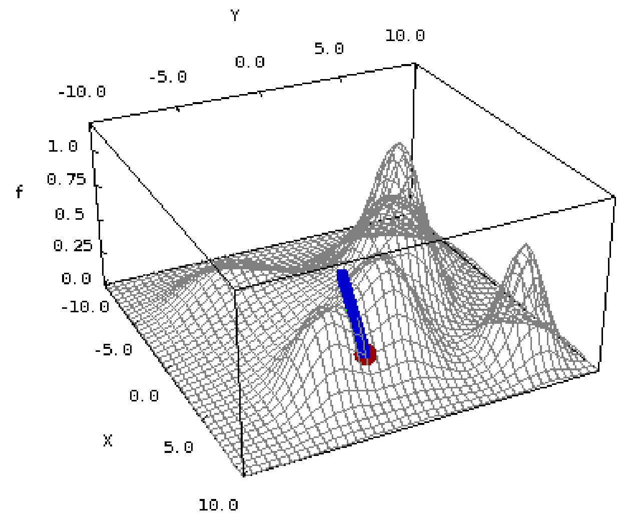
- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (divgrad)

These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Gradient – show the direction and size of the greatest change of a scalar field in each point of its domain,
input of the operation: scalar field
output of the operation: vector field

$$gradU = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$



where $\vec{i}, \vec{j}, \vec{k}$ are elementary vectors (pointing in the direction of each coordinate axis – see 2. lecture, 26.slide)

Comment to the notation:
We can write gradient using the so called nabla or del operator ∇:

$$gradU = \nabla U$$

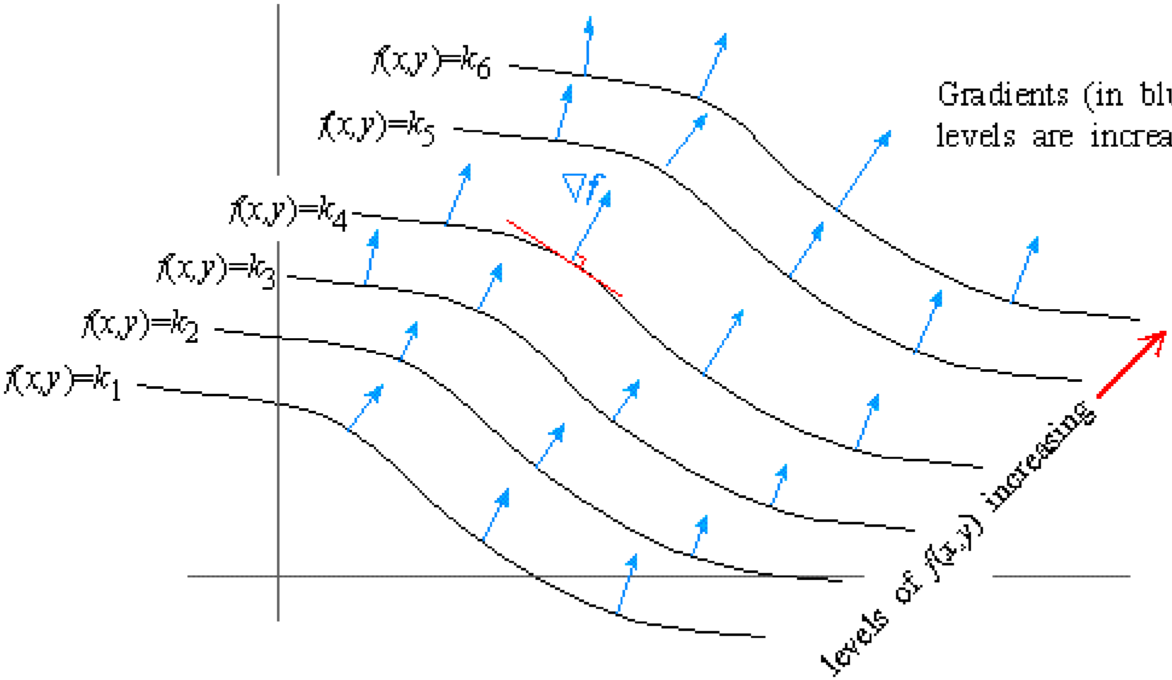
where

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Differential operators:

Gradient – show the direction and size of the greatest change of a scalar field in each point of its domain.

$$gradU = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$



In physical fields, gradient is always pointing in the direction of force lines (perpendicular to equipotential lines).

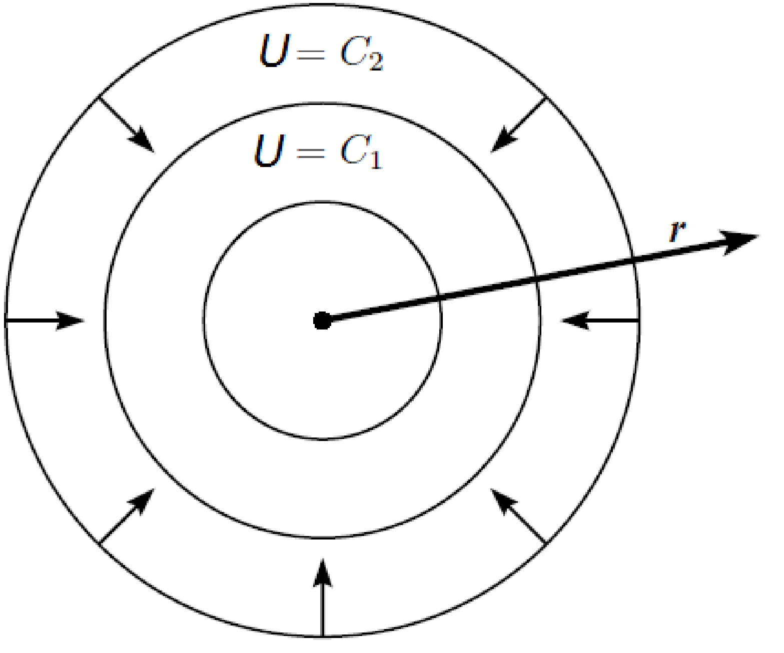
Gradient – example (field of positive electrical charge): (1/3)

Electrical potential *U*, caused by a positive electrical point charge (*Q*), situated in the origin of the coordinate system (Cartesian) can be described by means of the following equation:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2+y^2+z^2}}$$

where ε_0 is the electrical permitivity of vacuum ($8.854\cdot10^{-12}$ F/m).

Equipotential surfaces of this scalar field build spherical surfaces around the origin of the coordinate system. Gradient is a vector field, which vectors point in each point of the space perpendicular to these equipotential surfaces.



Gradient – example (field of positive electrical charge): (2/3)

$$gradU = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$

We will evaluate the gradient of this scalar function:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2 + y^2 + z^2}}$$

because the field of electrical intensity (vector) is given: $\vec{E} = -gradU$

First we evaluate the partial derivatives of *U* with respect to *x*, *y* and *z*.

$$\frac{\partial U}{\partial x} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left(\left[x^2 + y^2 + z^2\right]^{-1/2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\left(\left[x^2 + y^2 + z^2\right]^{-3/2} 2x\right) =$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{\left[x^2 + y^2 + z^2\right]^{3/2}}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right)$$

Partial derivatives $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$ are evaluated in a very similar way.

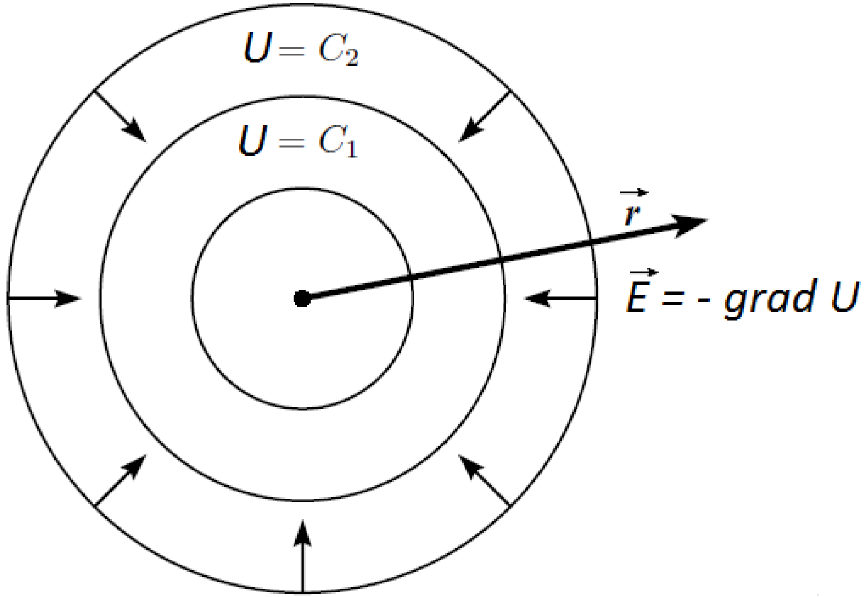
Gradient – example (field of positive electrical charge): (3/3)

$$\frac{\partial U}{\partial x} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right), \qquad \frac{\partial U}{\partial y} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{r^3}\right), \qquad \frac{\partial U}{\partial z} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

$$\vec{E} = -gradU = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\vec{i} + \frac{y}{r^3}\vec{j} + \frac{z}{r^3}\vec{k}\right) = \boxed{\frac{Q}{4\pi\varepsilon_0}\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r^3}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}}$$

This is a vector field, pointing in the same direction as the vector $\vec{r}$ and having the size:

$$\left|\vec{E}\right| = \frac{Q}{4\pi\varepsilon_0}\frac{r}{r^3} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$$



Differential operators:

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- divergence (div)
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- Laplacian operator (divgrad)

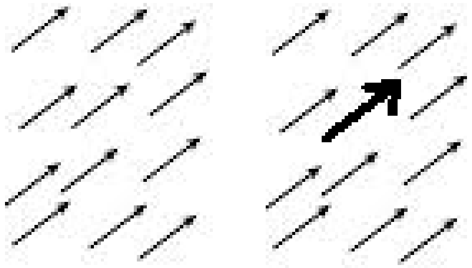
These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Divergence – tells about the sources of a vector field: when the result is zero then there is no source of the field in the point.

input of the operation: components of vector field

output of the operation: scalar value field



$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

where A_x, A_y, A_z are the components of vector $\vec{A}$.

Comment: Divergence depends on the changes of the size of vector components and not the change of their direction.

Comment to the notation:

We can write also divergence using the nabla or del operator ∇:

$$div\vec{A} = \nabla \cdot \vec{A}$$

where

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Divergence – example (field of electrical charge): (1/2)

Field of electrical intensity (a vector field) is given by:

$$\vec{E} = -gradU = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r^3}\right) = E_x\vec{i} + E_y\vec{j} + E_z\vec{k}$$

$$E_x = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right),\ \mathrm{E_y} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{r^3}\right),\ \mathrm{E_z} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

To evaluate the divergence of this field, we need to evaluate the following derivatives:

$$\frac{\partial \mathrm{E_x}}{\partial \mathrm{x}} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{x}}\left(\frac{x}{r^3}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{x}}\left(\frac{x}{\left[x^2+y^2+z^2\right]^{3/2}}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{x}}\left(x\left[x^2+y^2+z^2\right]^{-3/2}\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} + x\left(\frac{-3}{2}\right)\left[x^2+y^2+z^2\right]^{-5/2}2x\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3x^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

Divergence – example (field of electrical charge): (2/2)

For all three derivatives we get:

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3x^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3y^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3z^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left(3\left[x^2+y^2+z^2\right]^{-3/2} - 3\left(x^2+y^2+z^2\right)\left[x^2+y^2+z^2\right]^{-5/2}\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(3\left[x^2+y^2+z^2\right]^{-3/2} - 3\left[x^2+y^2+z^2\right]^{-3/2}\right) \boxed{= 0}$$

This result is valid for all points with the exception of the coordinate system origin, where x = y = z = 0 (source area).

Differential operators:

There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (divgrad)

These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Rotation – tells about the existence of so called curls of the vector field (not about the sources).

input of the operation: components of vector field

output of the operation: vector field

$$rot\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Comment: Rotation does not depend on the changes of the size of vector components (this was the role of divergence).

Comment to the notation:

We can write also divergence using the nabla or del operator ∇:

$$rot\vec{A} = \nabla \times \vec{A}$$

where

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Rotation – example (field of electrical charge): (1/2)

Field of electrical intensity (a vector field) is given by:

$$\vec{E} = -gradU = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r^3}\right) = E_x\vec{i} + E_y\vec{j} + E_z\vec{k}$$

$$E_x = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right),\ \mathrm{E_y} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{r^3}\right),\ \mathrm{E_z} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

For the rotation evaluation we need following derivatives:

$$\frac{\partial \mathrm{E_z}}{\partial \mathrm{y}} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{y}}\left(\frac{z}{r^3}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{y}}\left(\frac{z}{\left[x^2+y^2+z^2\right]^{3/2}}\right) = \frac{zQ}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{y}}\left(\left[x^2+y^2+z^2\right]^{-3/2}\right) =$$

$$= \frac{zQ}{4\pi\varepsilon_0}\left(\left(\frac{-3}{2}\right)\left[x^2+y^2+z^2\right]^{-5/2}2y\right) = \frac{-3yzQ}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial \mathrm{E_y}}{\partial \mathrm{z}} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{z}}\left(\frac{y}{\left[x^2+y^2+z^2\right]^{3/2}}\right) = \frac{-3zyQ}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

Rotation – example (field of electrical charge): (2/2)

From the evaluated derivatives it follows:

$$\frac{\partial \mathrm{E}_z}{\partial \mathrm{y}} - \frac{\partial \mathrm{E}_y}{\partial \mathrm{z}} = 0$$

In a similar way we can show:

$$\frac{\partial \mathrm{E}_x}{\partial \mathrm{z}} - \frac{\partial \mathrm{E}_z}{\partial \mathrm{x}} = 0 \qquad \frac{\partial \mathrm{E}_y}{\partial \mathrm{x}} - \frac{\partial \mathrm{E}_x}{\partial \mathrm{y}} = 0$$

... and for the rotation it is valid:

$$rot\vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \vec{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \vec{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \vec{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \boxed{= \vec{0}}$$

This result is valid for all points with the exception of the coordinate system origin, where x = y = z = 0 (source area).

Differential operators:

There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (divgrad)

These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Laplacian operator – in mathematical physics is often used the following (combined) differential operator,
input of the operation: scalar field
output of the operation: scalar field

$$div\left(gradU\right) = \frac{\partial\left(\partial U/\partial x\right)}{\partial x} + \frac{\partial\left(\partial U/\partial y\right)}{\partial y} + \frac{\partial\left(\partial U/\partial z\right)}{\partial z}$$

$$div\left(gradU\right) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Comment to the notation:
We can write gradient using the so called nabla or del operator ∇:

$$div\left(gradU\right) = \nabla \cdot \left(\nabla U\right) = \nabla^2 U = \Delta U$$

Differential operators:

Beside this combined operator (Laplacian), the are valid following equations:

$$rot\left(gradU\right)\equiv 0$$

$$div\left(rot\vec{A}\right)\equiv 0$$

These equations have important impacts on the properties of some physical fields:

- the first one tells that so called potential fields (which intensity can be expressed by means of the gradient) can not build curls,
- the second one tells us that in a curl there are no sources.

You can try to check it mathematically (make a proof) in a frame of a homework.

Lecture 9: functions of several variables

Content:

- basic definitions and properties
- partial and total differentiation
- differential operators
- multiple integrals
- examples of multiple integrals

Functions of several variables:

Multiple integrals (as „antipole" of partial derivatives):

The multiple integral is a generalization of the definite integral to functions of more than one real variable, for example, $f(x, y)$ or $f(x, y, z)$.
Integrals of a function of two variables over a region in R^2 are called double integrals, and integrals of a function of three variables over a region of R^3 are called triple integrals.
General form of a multiple integral:

$$\int \cdots \int_{\mathbf{D}} f(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n$$

The domain *D* of integration is either represented symbolically for every argument over each integral sign, or is abbreviated by a variable at the rightmost integral sign.

$$\int\limits_{x_1} \cdots \int\limits_{x_n} f(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n$$

Functions of several variables:

Multiple integrals:

Basic rule: the so called changing the order of integration (or reversing the order of integration).

In the case of a double integral: $\iint_D f(x,y)\,dA$

We can integrate with respect to x first: $\iint_D f(x,y)\,dA = \int_{\square}^{\square} \left(\int_{\square}^{\square} f(x,y)\,dx \right) dy,$

... or with respect to y first: $\iint_D f(x,y)\,dA = \int_{\square}^{\square} \left(\int_{\square}^{\square} f(x,y)\,dy \right) dx.$

We often say that the first integral is in *dxdy* order and the second integral is in *dydx* order.

Limits (bounds) of integration (boxes □) can be numbers and sometimes also functions.

Comment: In some situations, we know the limits of integration the *dxdy* order and need to determine the limits of integration for the equivalent integral in *dydx* order (or vice versa).

Functions of several variables:

Multiple integrals – simple example:

Let $R = [0, 2] \times [0, 1]$. Evaluate the double integral

$$\iint_R xe^y \, dA$$

using both possible orders of integration,
we can write this double integral as either of the iterated integrals

$$\int_0^2 \int_0^1 xe^y dy\, dx, \int_0^1 \int_0^2 xe^y dx\, dy.$$

The former integral is equal to

$$\int_0^2 xe^y \Big|_{y=0}^{y=1} dx = \int_0^2 x(e-1)\, dx = \frac{(e-1)x^2}{2}\Big|_0^2 = 2(e-1).$$

The latter integral is equal to

$$\int_0^1 e^y \frac{x^2}{2}\Big|_{x=0}^{x=2} dy = \int_0^1 2e^y\, dy = 2e^y\Big|_0^1 = 2(e-1).$$

As expected, these two iterated integrals are equal to each other.

Functions of several variables:

Multiple integrals – simple example:

Sometimes it is easier to integrate with respect to one variable first instead of the other variable. For example, let $R = [0, \pi] \times [0, 1]$, and evaluate the double integral

$$\iint_R x\cos(xy)\, dA.$$

Which variable is it easier to integrate with respect to first? If we want to integrate with respect to x, we will need to perform an integration by parts. However, if we integrate with respect to y, we need only use a quick u-substitution, $u = xy$. Then $du = x\, dy$, and we get

$$\iint_R x\cos(xy)\, dA = \int_0^\pi \int_0^1 x\cos(xy)\, dy\, dx = \int_0^\pi \left(\sin(xy)\Big|_{y=0}^{y=1} \right) dx =$$

$$= \int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = 2.$$

While in principle it does not matter which variable you integrate with respect to first, in practice it can be computationally easier to integrate with respect to one variable first instead of using the other variable.

Functions of several variables:

Multiple integrals – double integrals:

Properties of double integrals (valid also for triple, etc.):

- For two functions f and g over a region D,

$$\int\int_D [f(x,y)+g(x,y)]\,dxdy = \int\int_D f(x,y)dxdy + \int\int_D g(x,y)dxdy$$

- For constant c,

$$\int\int_D cf(x,y)dxdy = c\int\int_D f(x,y)dxdy.$$

- If $D = D_1 \bigcup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then

$$\int\int_D f(x,y)dxdy = \int\int_{D_1} f(x,y)dxdy + \int\int_{D_2} f(x,y)dxdy$$

Multiple integrals – double integrals over general regions:

A plane region D is said to be of type I if

$$D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\},$$

where g_1 and g_2 are continuous on $[a, b]$.

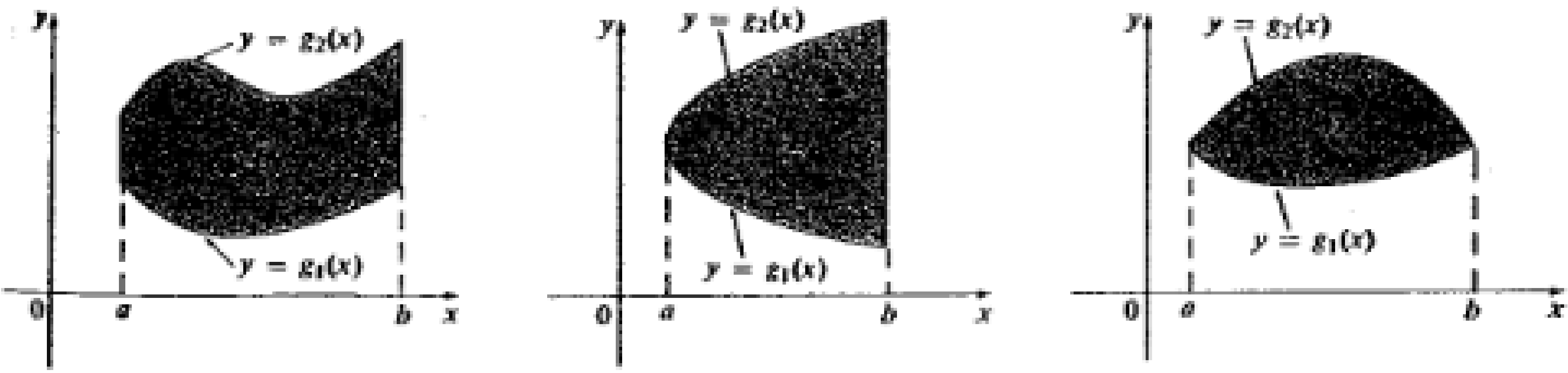


Figure Some type I regions

If f is continuous on a type I region

$$D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\},$$

then

$$\int\int_D f(x,y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx$$

Multiple integrals – double integrals over general regions:

Example . Evaluate $\int\int_D (x+2y)dxdy$, where

$$D = \{(x,y) | -1 \le x \le 1, 2x^2 \le y \le 1 + x^2\}.$$

Solution. Note that D is a region of type I. We have

$$\begin{aligned}
\int\int_D (x+2y)dxdy &= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)dydx \\
&= \int_{-1}^{1}\left[\int_{2x^2}^{1+x^2} (x+2y)dy\right] dx \\
&= \int_{-1}^{1} (xy+y^2)\Big|_{2x^2}^{1+x^2} dx \\
&= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2 - (x \cdot 2x^2 + 4x^4)\right] dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx \\
&= \left[-\frac{3}{5}x^5 - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\right]\Bigg|_{-1}^{1} = \frac{32}{15}.
\end{aligned}$$

Double integrals - example (so called Gaussian integral): (1/3)

In lecture nr.9 (slide nr. 6) we have mentioned that solutions of some indefinite integrals do not exist, when we describe the primitive functions by means of elementary functions.

One of this functions was also the exp($-x^2$), used often in statistics. But in the case of an unbounded (improper) integral, the solution can be found by means of a double integral.

So, we try to find the solution of the following (Gaussian) integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

We can formulate a square of the searched integral I:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

where the dummy variable y has been substituted for x in the last integral. This is now a double integral, which can be rewritten:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

The product of two integrals can be expressed as a double integral:

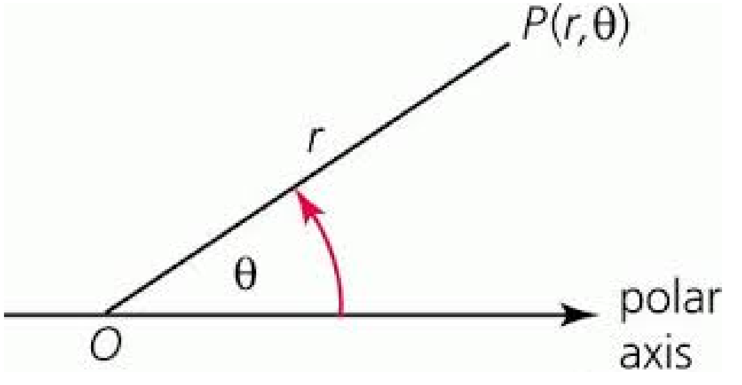
$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

The differential *dxdy* represents an element of area in Cartesian coordinates. An alternative representation of the last integral can be expressed in plane polar coordinates *r*, *θ*.

These two coordinate systems are related by following relations:

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

The element of area in polar coordinates is given by *rdrdθ* (exactly: *dr·rdθ*), so that the double integral becomes:



$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$I^2 = \int_0^\infty \int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta$$

Integration over θ gives a factor 2π. The integral over r can be done after the substitution $u = r^2$, $du = 2rdr$:

$$I^2 = 2\pi \int_0^\infty e^{-r^2}\, r\, dr = 2\pi \tfrac{1}{2} \int_0^\infty e^{-u}\, du = \pi$$

Finally, we can write:

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}}$$

This nice and simple solution we were not able to obtain by means of the solution of indefinite integral of one variable...

Double integrals – next examples:

Using again a double integral in plane polar coordinates (*r*, *θ),*
we can write (element of area in polar coordinates is given by *rdrdθ*):

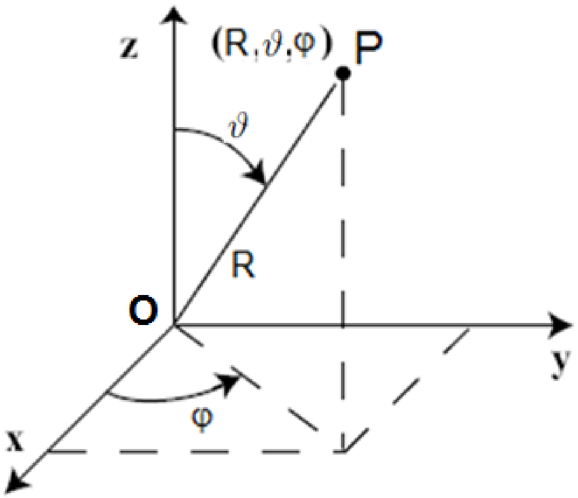
$$S = \int_0^R \int_0^{2\pi} r dr d\theta = 2\pi \int_0^R r dr = 2\pi \left[\frac{r^2}{2} \right]_0^R = \boxed{\pi R^2}$$

This gave us the well known formula for circle area evaluation.

Coming back to the spherical coordinate system (R, ϑ, φ):

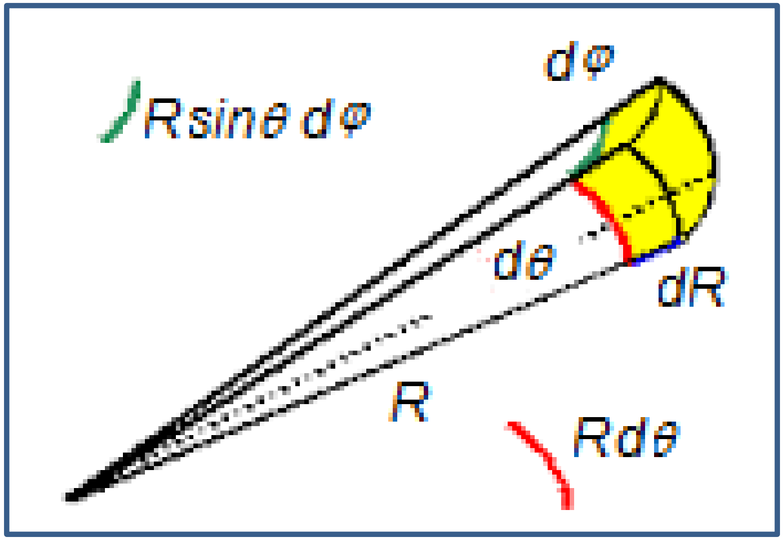
$$S = \int_0^{\pi} \int_0^{2\pi} R^2 \sin\vartheta d\vartheta d\varphi = 2\pi R^2 \int_0^{\pi} \sin\vartheta d\vartheta =$$

$$= 2\pi R^2 [-\cos\vartheta]_0^{\pi} = 2\pi R^2 [1+1] = \boxed{4\pi R^2}$$



This gave us the well known formula for a sphere surface evaluation.

Triple integral – example:



Staying with the spherical coordinate system (R, ϑ, φ):

$$V = \int_0^{\pi}\int_0^{2\pi}\int_0^{a} R^2 \sin\vartheta d\vartheta d\varphi dR = 2\pi \int_0^{\pi}\int_0^{a} R^2 \sin\vartheta d\vartheta dR =$$

$$= 2\pi \int_0^{a} \left[-\cos\vartheta\right]_0^{\pi} R^2 dR = 4\pi \int_0^{a} R^2 dR =$$

$$= 4\pi \left[\frac{R^3}{3}\right]_0^{a} = \boxed{\frac{4}{3}\pi a^3}$$

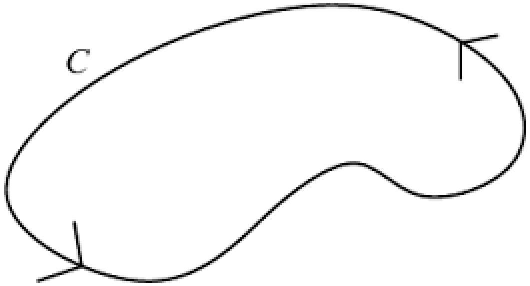
This gave us the well known formula for a sphere volume evaluation.

Integrals – additional comments:

In various text-books and scientific papers you can find notation of integrals with a circle crossing the symbols of integrals. These show integrals evaluated along closed curves or surfaces.

$$\oint_C A\,dl$$

- so called circulation integral (curve integral - along the closed curve *C*), *A* is the integrated function, *dl* is the element of the curve



$$\oiint_S E\,ds$$

- so called flux integral (surface integral - over the closed surface *S*), *E* is the integrated function, *ds* is the element of the surface