Lecture 3: Oscillators, waves

Content:

- harmonic oscillator
- mathematical pendulum
- damped oscillator
- driven oscillator
- waves
- Huygens principle, Doppler effect

HARMONIC OSCILLATOR

- **Simple harmonic oscillator -** F is the only acting force
- **Damped oscillator –** Friction (damping) occurs
- **Driven oscillator –** Damped oscillator further affected by external force

Simple harmonic oscillator

- balance of the system is given by (with help of Newton's second law)

$$F = ma = m\frac{d^2x(t)}{dt^2} = -kx(t)$$

Homogenous LDE with constant coefficients

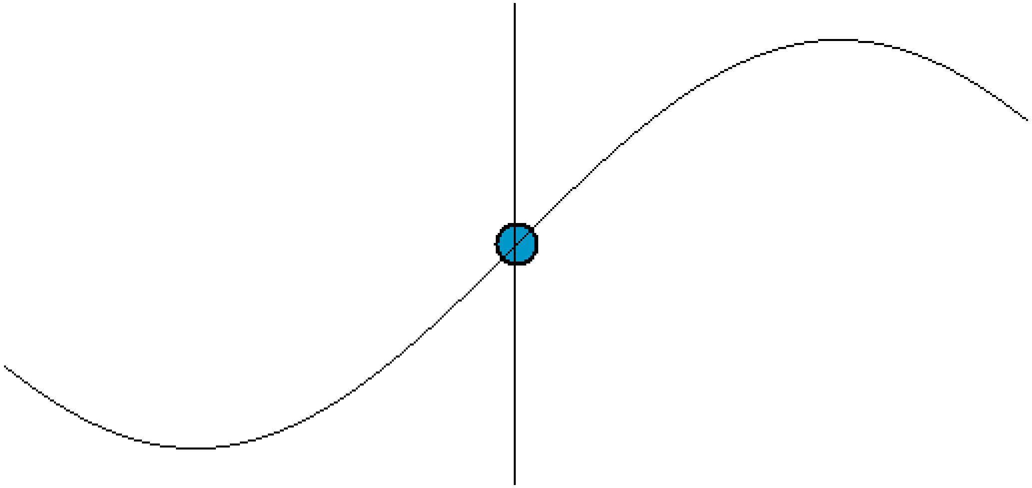
$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t) = A\cos(\omega t - \varphi)$$

Periodic motion

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi f$$

$$A = \sqrt{C_1^2 + C_2^2}$$

$$\tan\varphi = \frac{C_1}{C_2}$$



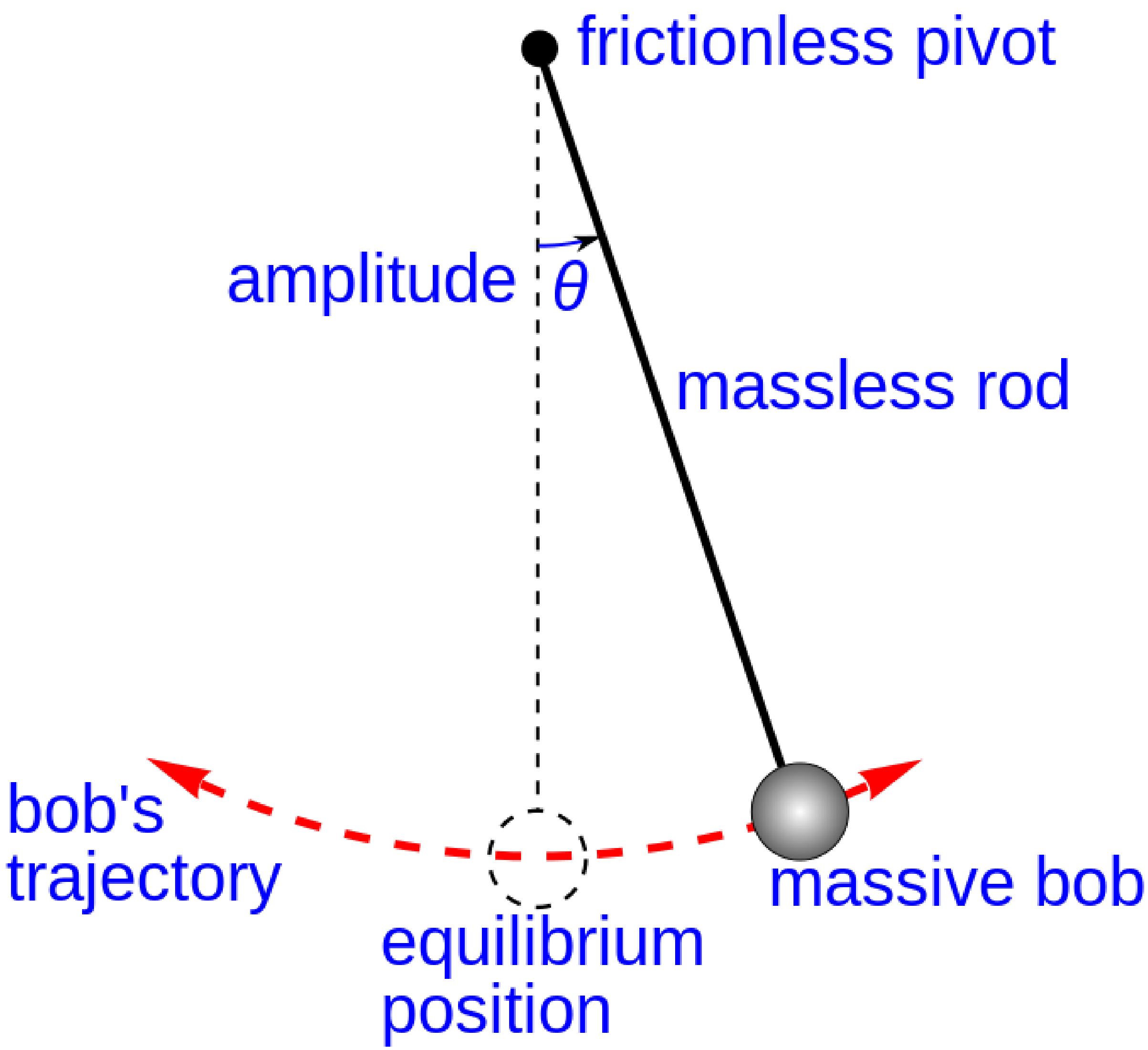
Simple harmonic oscillator

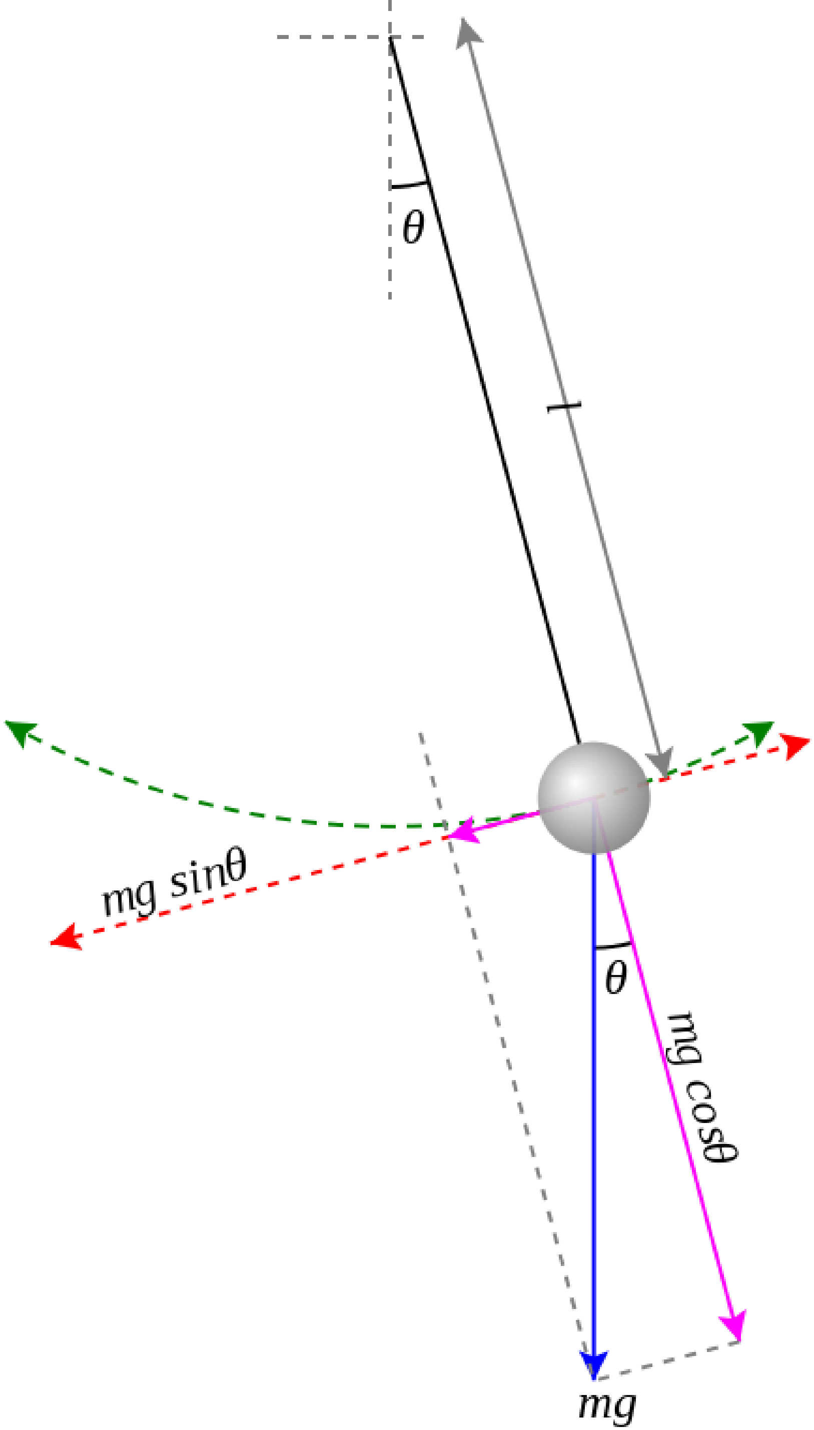
Velocity $$\vec{v}(t) = \frac{dx(t)}{dt} = -A\omega\sin(\omega t - \varphi)$$

Speed $$\left|\vec{v}(t)\right| = \omega\sqrt{A^2 - x^2(t)}$$

Acceleration $$\vec{a}(t) = \frac{d^2x(t)}{dt^2} = -A\omega^2\cos(\omega t - \varphi)$$

Example – simple gravity pendulum





simple gravity pendulum

$$F = m \cdot a$$

$$F = -mg\sin\theta = m \cdot a \Rightarrow a = -g\sin\theta$$

$$s = l \cdot \theta \rightarrow v = \frac{ds}{dt} = l\frac{d\theta}{dt} \rightarrow a = l\frac{d^2\theta}{dt^2}$$

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

assumption: θ << 1 $\rightarrow$ **sin θ ≈ θ**

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

boundary conditions $\theta(0) = \theta_0; \quad \left.\frac{d\theta}{dt}\right|_{t=0} = 0$

Solution $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$ **θ_0 << 1**

Period $T = 2\pi\sqrt{\frac{l}{g}}$

Damped oscillator

Friction force $F_f = -cv$

$$F = F_{ext.} - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$ if $F_{ext.} = 0$

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}};\ \zeta = \frac{c}{2\sqrt{mk}}$$

$\zeta > 1$ overdamped $\qquad \zeta = 1$ critically damped $\qquad \zeta < 1$ underdamped

$$x(t) = Ae^{-\zeta\omega_0 t}\sin\left(\sqrt{1-\zeta^2}\,\omega_0 t + \varphi\right)$$

Driven oscillator

Externally applied force $F(t)$

$$F = F_{ext.} - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2 x = \frac{F(t)}{m}$$

Solutions depend on external force

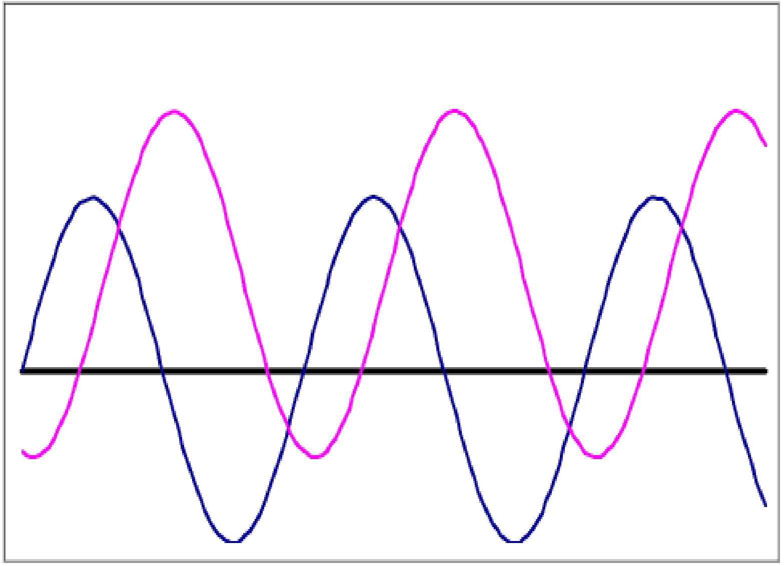
Combinations of oscillations

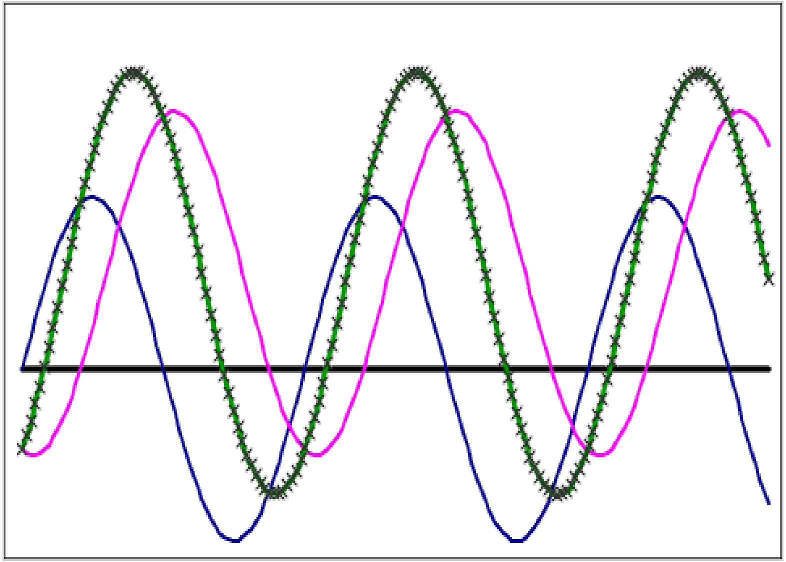
Oscillation 1 $x_1(t) = A_1 \cos(\omega_1 t - \varphi_1)$

Oscillation 2 $x_2(t) = A_2 \cos(\omega_2 t - \varphi_2)$

$$x = x_1 + x_2 = A_1 \cos(\omega_1 t - \varphi_1) + A_2 \cos(\omega_2 t - \varphi_2)$$

Maximum possible displacement $x_{max} = A_1 + A_2$



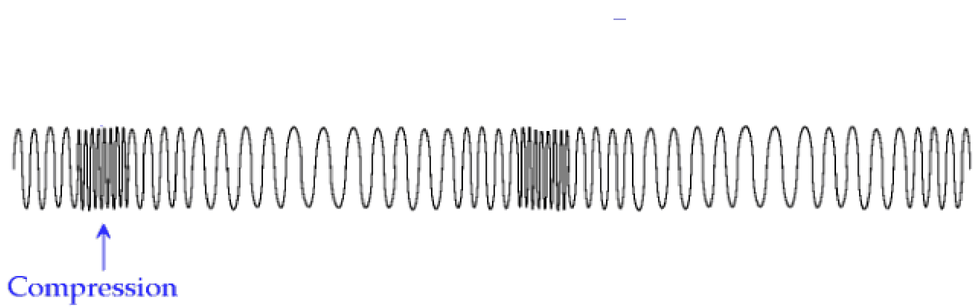


Waves

– oscillation accompanied by the transfer of energy which travels through mass or space
– little or no associated mass transport

waves
- mechanical – through medium which is deformed, e.g. sound waves
- electromagnetic – no medium, periodic oscillations of electric and magnetic fields, e.g. visible light

waves
- transverse – oscillations are perpendicular to the energy transfer
- longitudinal – oscillations are parallel to the energy transfer



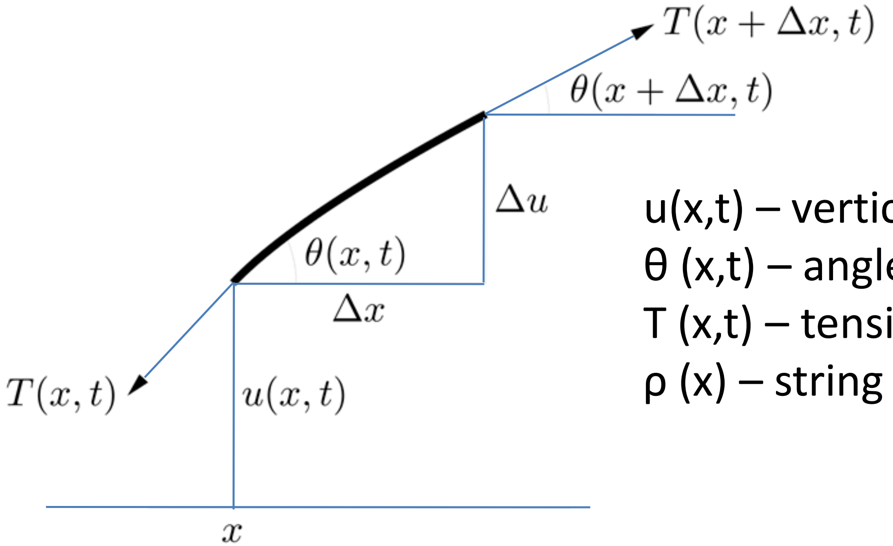
Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

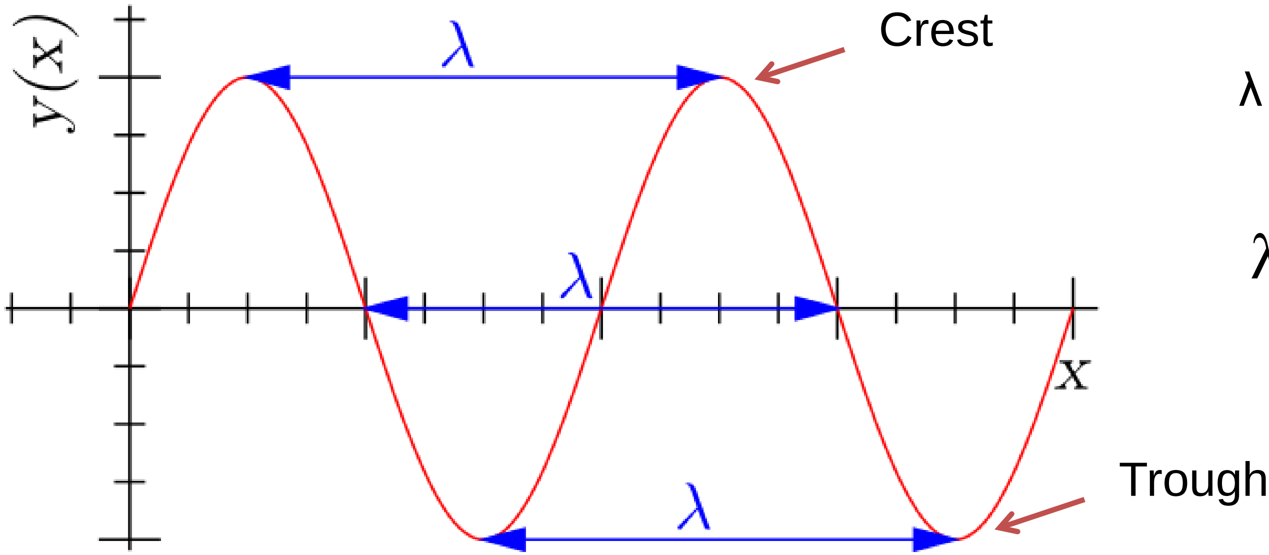
$u = u(x_1, x_2, ..., x_n, t)$ - scalar function whose values could model, for example, the mechanical displacement of a wave

one space dimension case: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Derivation based on Newton's law applied to an elastic string

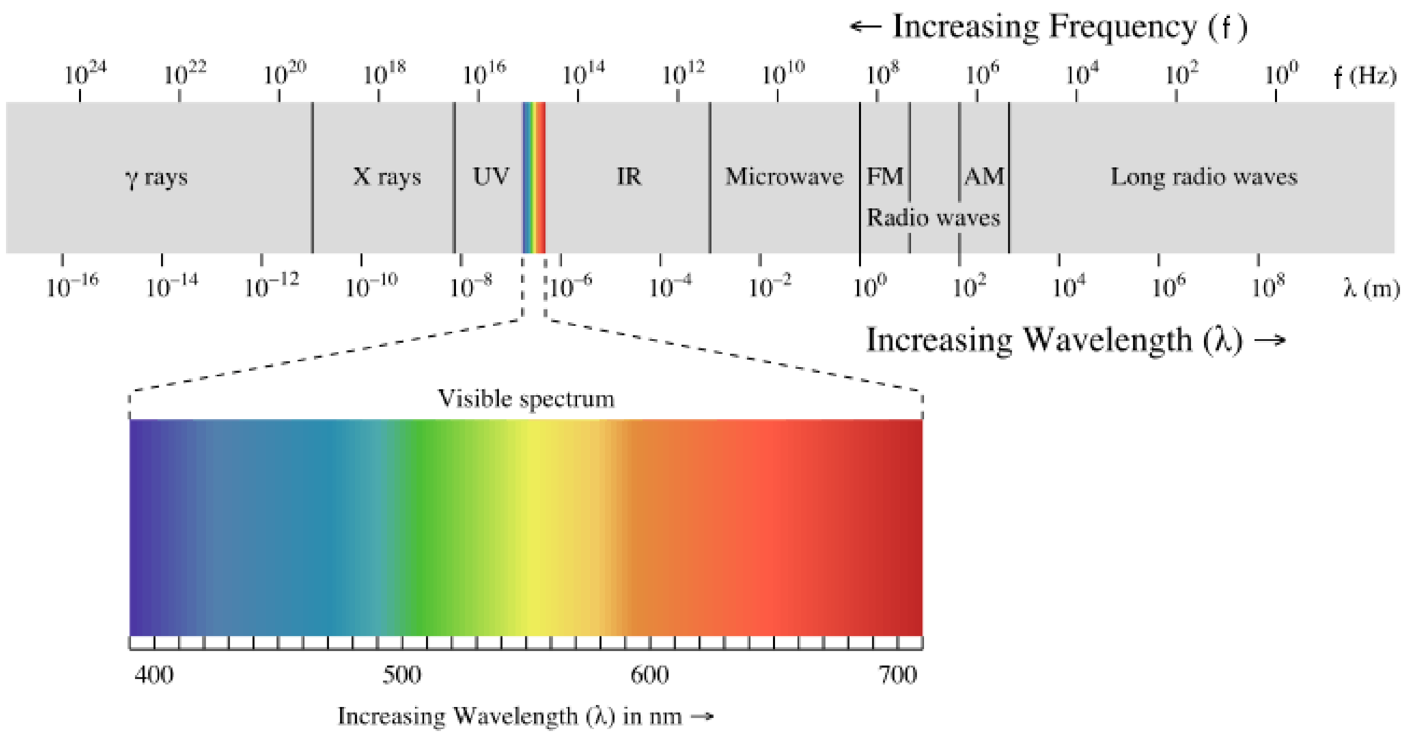


$u(x,t)$ – vertical displacement of the string from x axis
$\theta\ (x,t)$ – angle between the string and horizontal line
$T\ (x,t)$ – tension in the string
$\rho\ (x)$ – string mass density



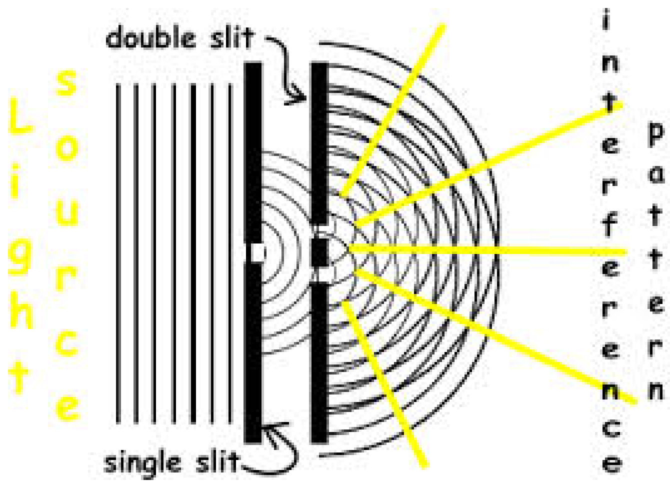
λ - wavelength

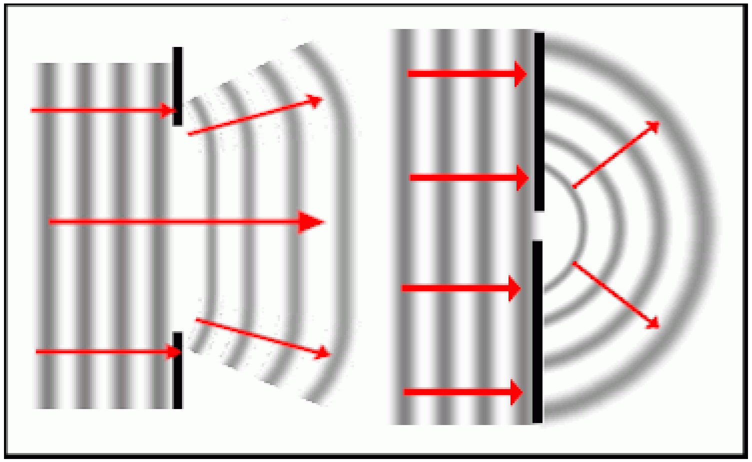
$$\lambda = vT = \frac{v}{f}$$



Huygens principle

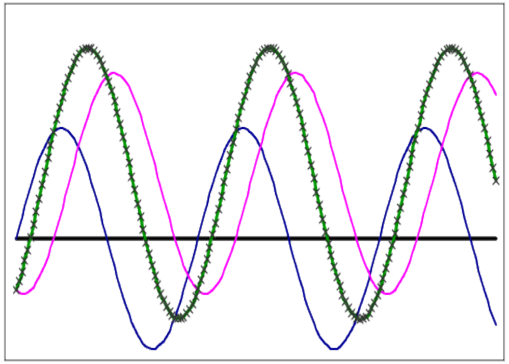
Every point on a wave-front may be considered a source of secondary spherical wavelets which spread out in the forward direction. The new wave-front is the tangential surface to all of these secondary wavelets.

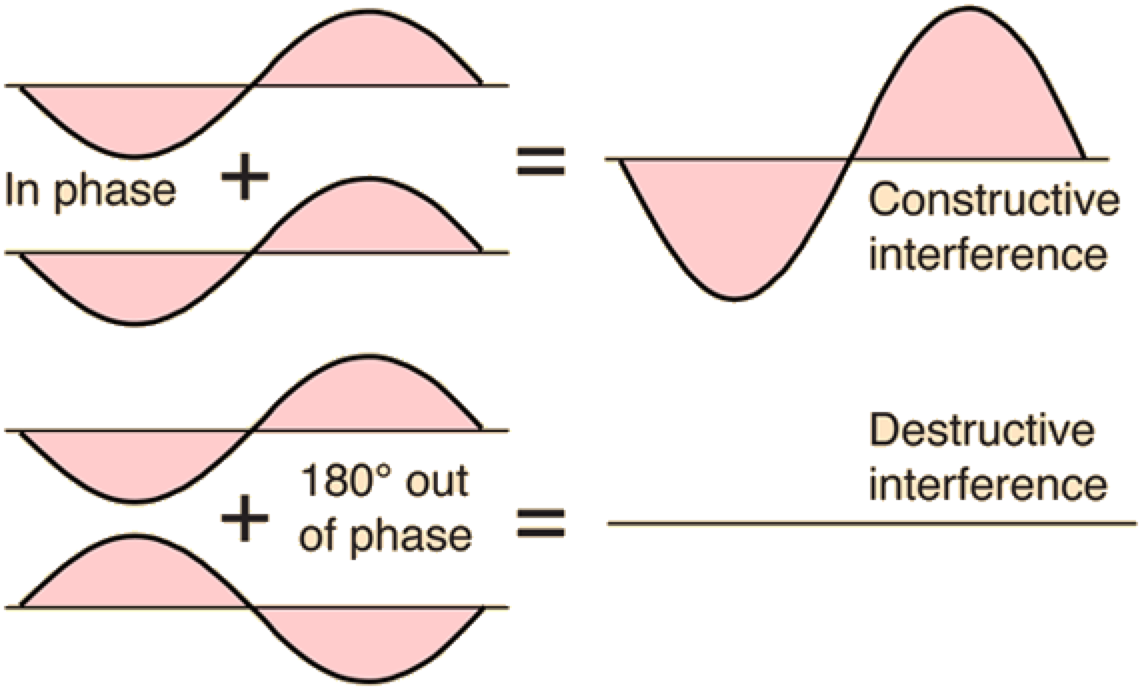




Interference

The principle of superposition of waves states that when two or more propagating waves of same type are incident on the same point, the total displacement at that point is equal to the point wise sum of the displacements of the individual waves.





Doppler effect

is the change in frequency of a wave (or other periodic event) for an observer moving relative to its source.