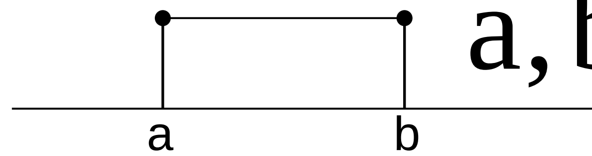
Lecture 3: Intervals, elementary functions and their properties

Content

- Intervals as sets of real numbers
- General properties of the functions
- Elementary functions

Intervals

closed interval: $\langle a, b \rangle$

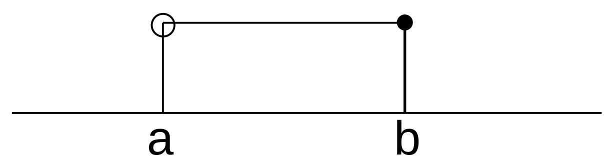


$a, b \in I$

open interval: (a, b)

$a, b \notin I$

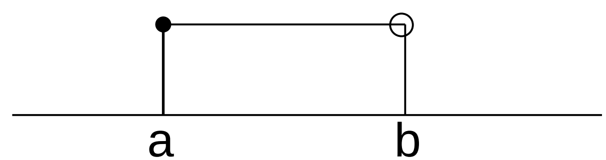
left-open interval: $(a, b\rangle$



$a \notin I$

$b \in I$

right-open interval: $\langle a, b)$

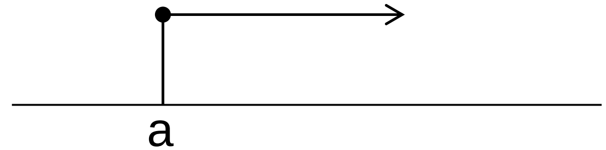


$a \in I$

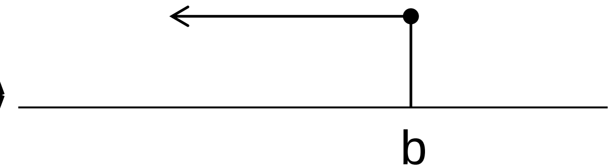
$b \notin I$

if $a \vee b \to \infty$

$$\langle a, \infty)$$

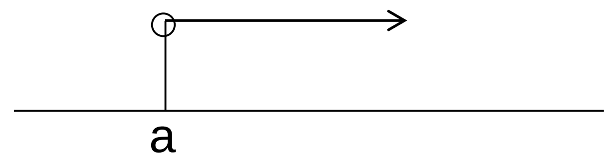


$$(-\infty, b\rangle$$

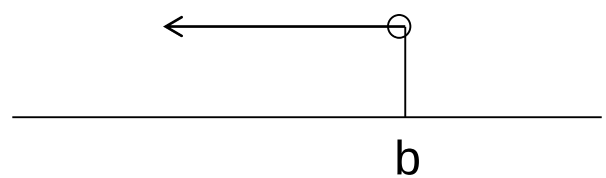


or

$$(a, \infty)$$



$$(-\infty, b)$$



Function

Definition $$f : M \rightarrow N, \forall x \in M \exists ! y \in M$$

In words: Let M and N be sets of numbers. The mapping f from M to N is called function if for **each x** from M exists **exactly one y** from N.

Note to symbolism: $\forall$ - for each, for all

$\exists$ - exist

! - exactly one, one and only one

Basic properties of the functions

$$f: M \to N, \forall x \in M \exists ! y \in M$$

M – domain of definition, signed D (f) – set of all x for which the function is defined (set where the function has a sense)

Expected question: where/when the function has no sense?

Answer: dividing by zero, even degree roots of negative numbers, logarithm of non-positive numbers…

N – range/image of function, signed H(f) – set of all possible results y

Important: domain of definition has to be the first thing you do before solving anything. It can not be changed in solving process.

$$f : M \to N, \forall x \in M \exists ! y \in M$$

x – independent variable, origin, co-image, pre-image

y – image (usually signed as f(x), too)

Equality of functions: two functions f and g are equal only if D(f(x)) = D(g(x)) and

$f(x) = g(x) \; \forall x \in D$

Example

Equal functions

$f(x) = \sqrt{x^2}$ $\quad g(x) = |x|$

$D(f) = \hat{A}$ $\quad D(g) = \hat{A}$

Unequal functions

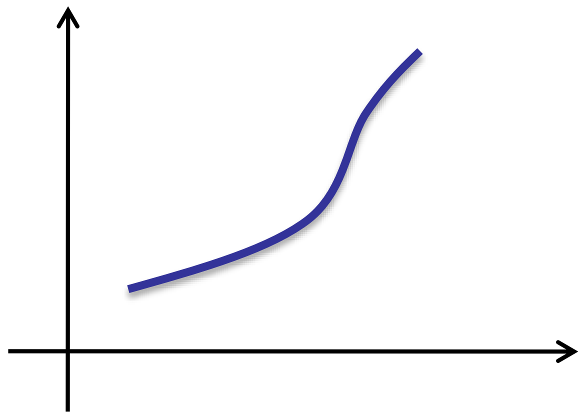
$$f(x) = \frac{(x-3)}{(x-3)(x+3)} \qquad g(x) = \frac{1}{(x+3)}$$

$$D(f) = \hat{A} - \{-3, 3\} \qquad D(g) = \hat{A} - \{-3\}$$

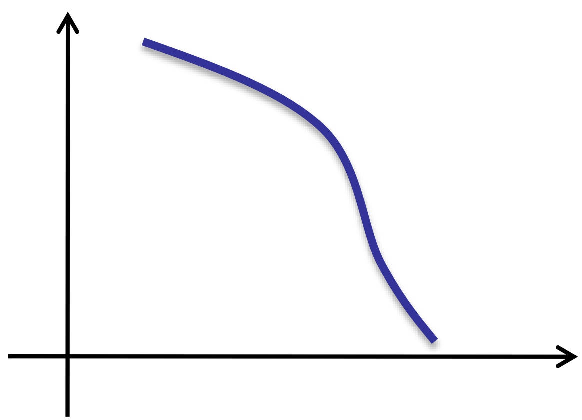
Monotonicity of the functions

- function is called monotonic if and only if is either increasing or decreasing on D(f)

Increasing function: $\forall x_1, x_2 \in D(f) \ | x_1 < x_2 : f(x_1) < f(x_2)$

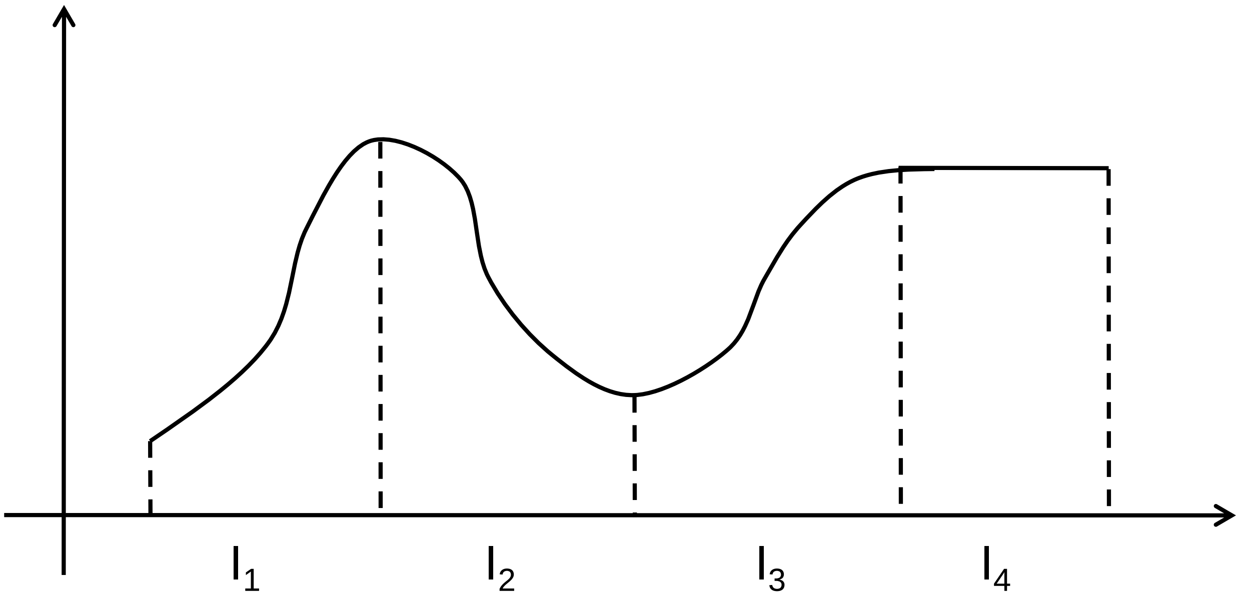


Decreasing function: $x_1, x_2 \in D(f) \ | x_1 < x_2 : f(x_1) > f(x_2)$



Intervals of monotonicity

If the function is not monotonic i.e. is not only increasing or decreasing on D(f), the D(f) can be divided into set of intervals on which the function is monotonic or constant



I_1 - increasing
I_2 - decreasing
I_3 - increasing
I_4 - constant

Injective function, one-to-one function

$$f : M \to N, \forall x_1, x_2 \in M \mid \; x_1 \neq x_2; f(x_1) \neq f(x_2)$$

- close relation with monotonicity
- increasing and decreasing functions are injective
- if the function is not injective, we can try to split the D(f) into set of intervals, where the function will be injective (same process as in case of the intervals of monotonicity)

Extrema

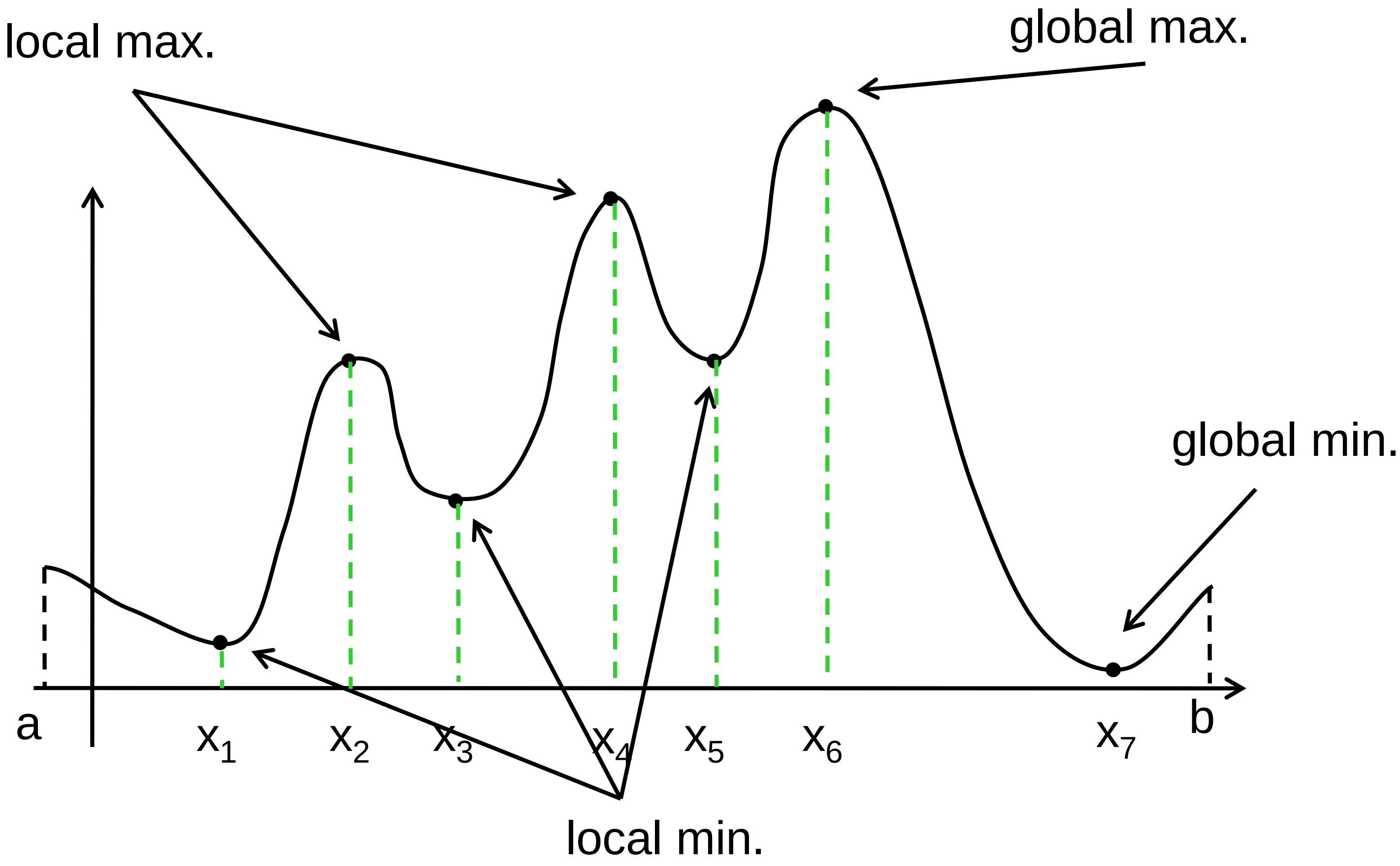
The function f defined on D(f) has global maximum in point x_0 if $\forall x \in D(f); f(x) < f(x_0)$

The function f defined on D(f) has global minimum in point x_0 if $\forall x \in D(f); f(x) > f(x_0)$

The function f defined on D(f) has local maximum in point x_0 if $\forall x \in A \subset D(f); f(x) < f(x_0)$

The function f defined on D(f) has local minimum in point x_0 if $\forall x \in A \subset D(f); f(x) > f(x_0)$

Example



local max.
global max.
global min.
a
x_1
x_2
x_3
x_4
x_5
x_6
x_7
b
local min.

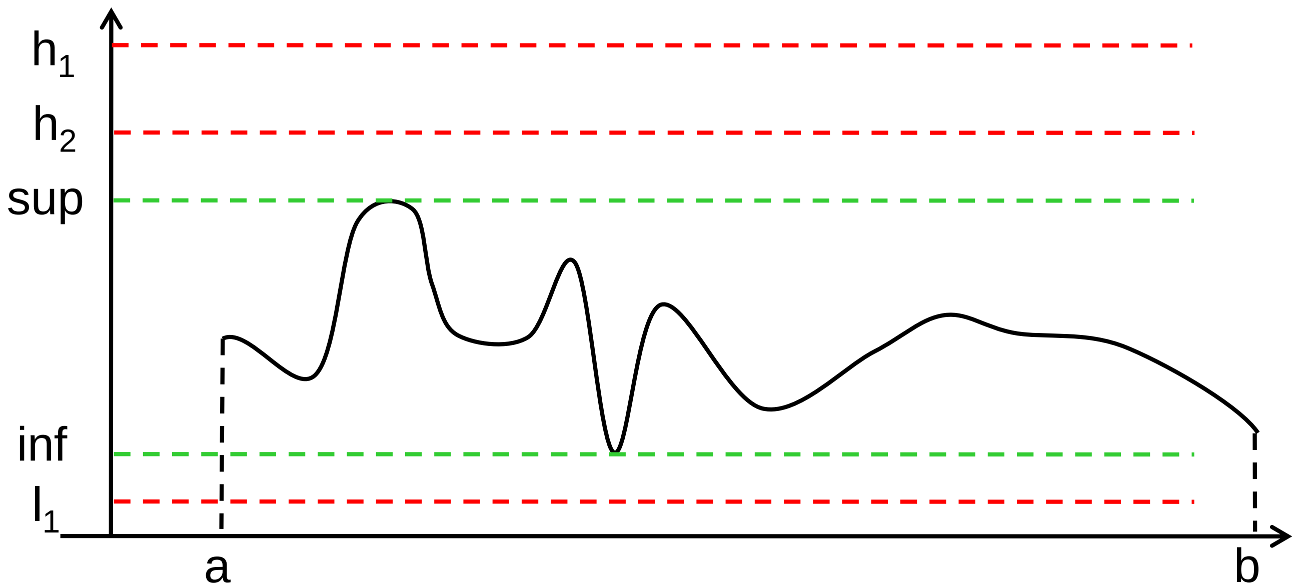
Bounded function, supremum, infimum

It is said the function has the upper boundary (or it is bounded from top), if there exist the number h such that $h \geq f(x)\ \forall x \in D(f)$

Supremum of the function f is the least number that is greater or equal of all elements from range (H(f)) of the function f (the lowest of all numbers h).

It is said the function has the lower boundary (or it is bounded from bottom), if there exist the number l such that $l \leq f(x)\ \forall x \in D(f)$

Infimum of the function f is the greatest number that is lower or equal of all elements from range (H(f)) of the function f (the greatest of all numbers l).



Function composition

Simply speaking – the composition of two (or more) functions f and g is function $h(x) = f \circ g = f(g(x))$, i.e. function g is the argument of function f.

The notation $f \circ g$ is read as "f circle g" or "g round f" etc.

Generally $f \circ g \neq g \circ f$

Example

f = x^2 and g = x+2

The composition h(x) = $f \circ g$ = $(x + 2)^2$ → $h(x) \neq k(x)$
The composition k(x) = $g \circ f$ = $x^2 + 2$

The decomposition of the function will be understand as process of splitting the function composition into single elements

Example

$$h(x) = \sqrt{\sin(x+5)} \rightarrow h(x) = \sqrt{p(x)}$$

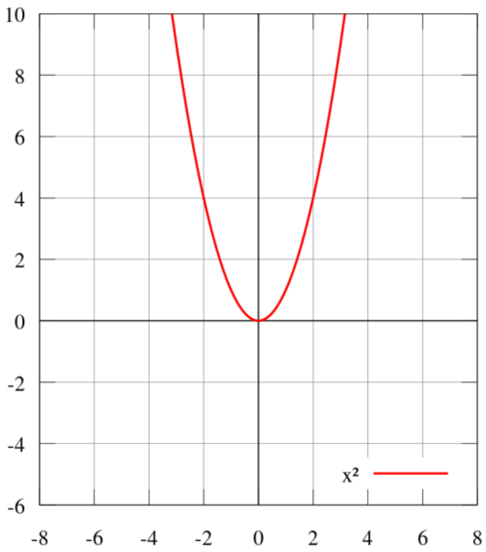
$$p(x) = \sin(q(x))$$

$$q(x) = x + 5$$

Even and odd functions

The function is named as even if it satisfies following condition $\left(\forall x \in D(f)\right)$:

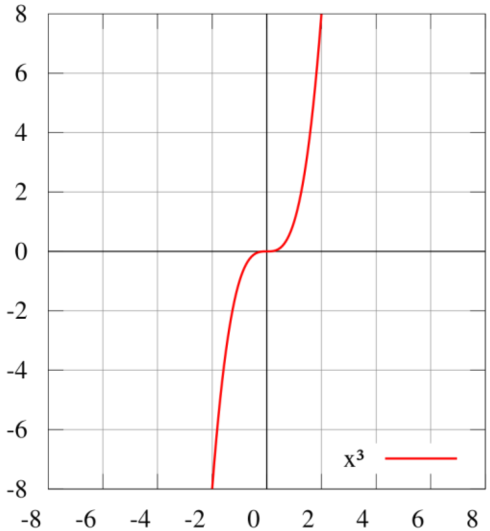
$$f(x) = f(-x)$$



consequence – the graph of even function is symmetrical around y axis

The function is named as odd if it satisfies following condition $\left(\forall x \in D(f)\right)$:

$$f(x) = -f(-x)$$



consequence – the graph of even function is symmetrical around origin

Inverse function

The function is named as inverse (signed $f^{-1}(x)$) if it satisfies following condition $(\forall x \in D(f))$:

$$f(x) \circ f^{-1}(x) = x$$

Example

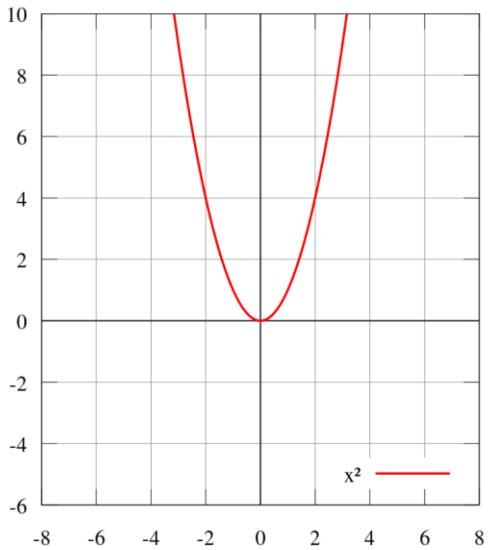
$f(x) \rightarrow y = \frac{5x-3}{2}$ $D(f) = R, H(f) = R$ $f^{-1}(x) \rightarrow y = \frac{2x+3}{5}$ $D(f^{-1}) = R, H(f^{-1}) = R$

$$f(x) \circ f^{-1}(x) = f(f(x)) = \frac{5\left(\frac{2x+3}{5}\right)-3}{2} = x$$

Important: The inverse function can be sought only if original function is **injection (one-by-one function).** If it is not injection, the D(f) must be divided into intervals where the function f is injection.

Example

$$f(x) \to y = x^2 \qquad D(f) = R,\ H(f) = \langle 0, \infty)$$



This function is not injection – two different inputs (e.g. x_1 = -2 and x_2 = 2) have the same image ($y_1 = y_2 = 4$).

The function f has to be divided into two parts with different D(f) on which the functions are injection:

$$f_1(x) \to y = x^2 \qquad D(f_1) = (-\infty, 0),\ H(f_1) = (0, \infty)$$

$$f_2(x) \to y = x^2 \qquad D(f_2) = \langle 0, \infty),\ H(f_2) = \langle 0, \infty)$$

Now we are ready to set the inverse functions:

$$f_1^{-1}(x) \to y = -\sqrt{x} \qquad D(f_1^{-1}) = (0, \infty),\ H(f_1^{-1}) = (-\infty, 0)$$

$$f_2^{-1}(x) \to y = \sqrt{x} \qquad D(f_2^{-1}) = \langle 0, \infty),\ H(f_2^{-1}) = \langle 0, \infty)$$

Important relationship: $D(f) = H(f^{-1}) \quad H(f) = D(f^{-1})$

How to find the inverse function

Assume that original function fulfilled the required condition (it is injection). Then the following algorithm can be used to find the inverse function:

1. D(f), D(f) = H(f^{-1})
2. Switch “y” and “x” symbols
3. Rearrange the function to y = f(x) to obtain inverse function
4. D(f^{-1}), D(f^{-1}) = H(f)

Example

$$f(x) \rightarrow y = \frac{1}{x-3}$$

1. $D(f) = \mathrm{R} - \{3\} \rightarrow H\left(f^{-1}\right) = \mathrm{R} - \{3\}$

2. "Switch" $\quad x = \dfrac{1}{y-3}$

3. "Rearrangment" $\quad x = \dfrac{1}{y-3} \Rightarrow y-3 = \dfrac{1}{x} \rightarrow \boxed{y = \dfrac{1}{x} + 3} = f^{-1}(x)$

4. $D\left(f^{-1}\right) = \mathrm{R} - \{0\} \rightarrow H(f) = \mathrm{R} - \{0\}$

Periodic function

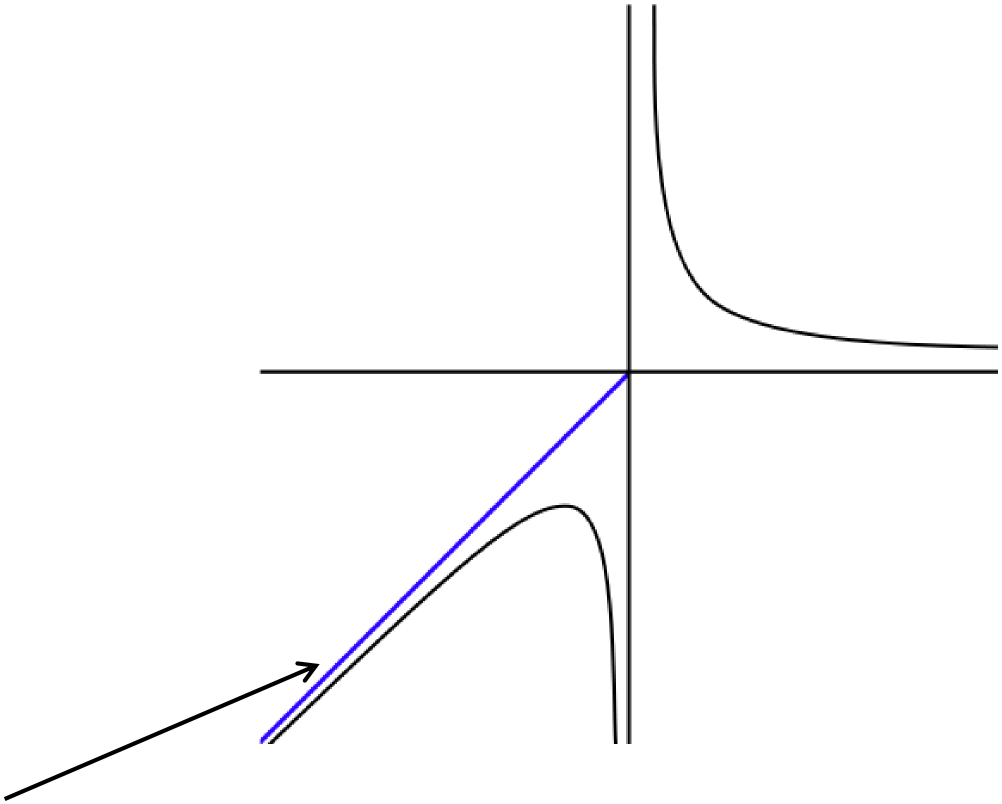
The function is named as periodic if it satisfies following condition $(\forall x \in D(f))$:

$$f(x + T) = f(x)$$

T – period of the function

Graph

Graphical representation of the collection of all ordered pairs [x, f(x)]



Asymptote: line such that the distance between the curve and the line approaches zero as they tend to infinity (the graph of function does not cross it, nor touch it)

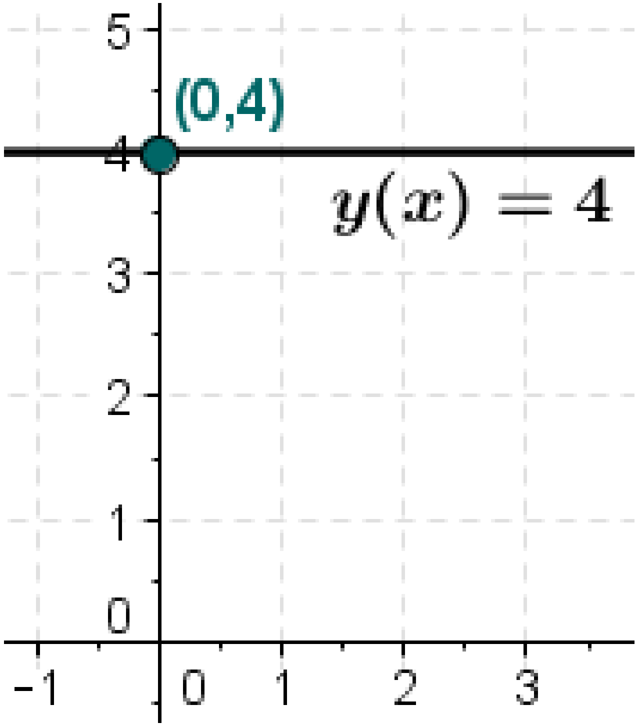
Elementary functions

Constant function

$$y = c$$

c is the constant, $c \in R$

$D(f) = R,\ H(f) = c$



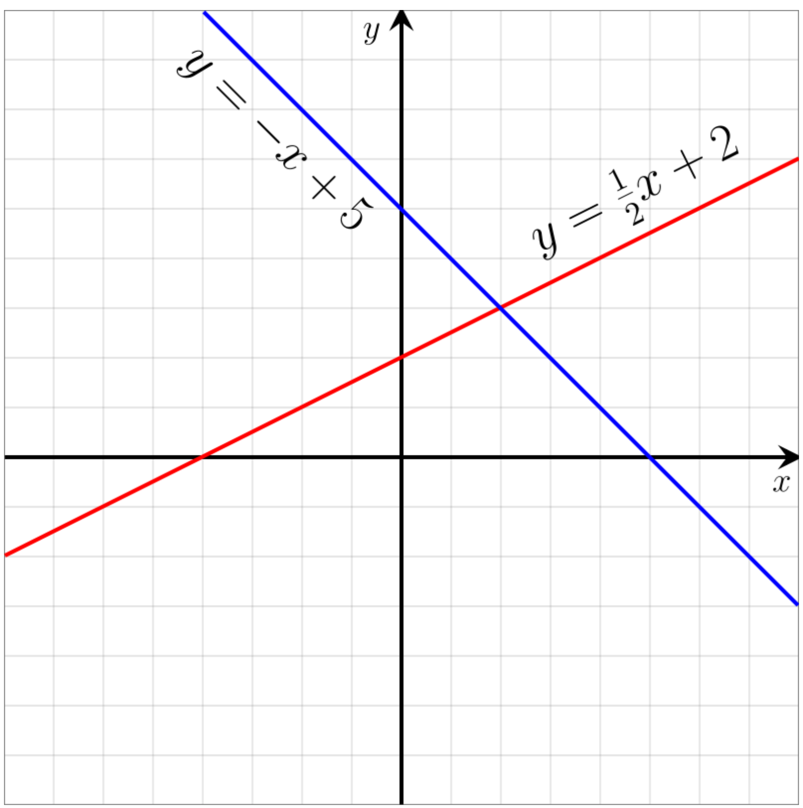
- even function
- not injection
- sup = inf = c

Linear function

$$y = kx + q$$

k, q are the constant, $k, q \in R$

graph is the straight line
"k" – tangent or linear term, controls the slope of line
"q" – absolute term, controls position of the line in direction of axis y
sign of term "k" controls the orientation of line: "+" for increasing line
"-" for decreasing line



$D(f) = R$, $H(f) = R$

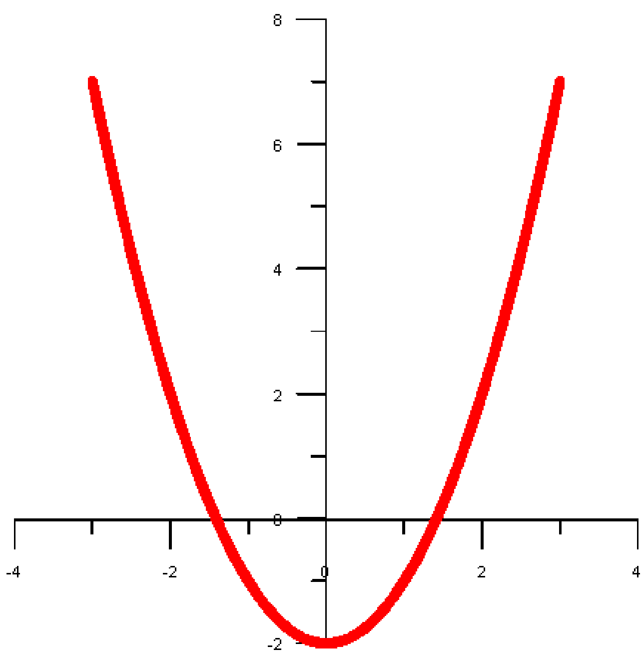
- if $q = 0 \rightarrow$ odd function
- injection
- sup, inf, max, min $\nexists$

Power function

$$y = ax^{b} + c \qquad a, c \in R,\ b \in Z \cup (0,1)$$

1. If b is positive even integer, the graph is parabola. The “a” term controls the slope of “parabola’s legs”. Sign of “a” controls orientation of parabola:
   if a > 0 – “hole” type
   if a < 0 – “hill” type
   “c” controls position of the line in direction of axis y

$y = x^2 - 2$

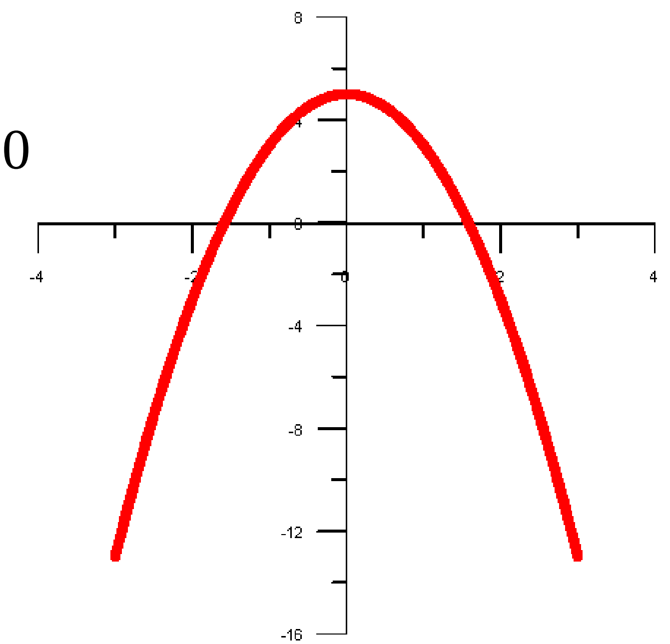


$$D(f) = R,$$

$$H(f) = \langle c, \infty), \text{ if } a > 0$$

$$H(f) = (-\infty, c\rangle, \text{ if } a < 0$$

$y = -2x^2 + 5$

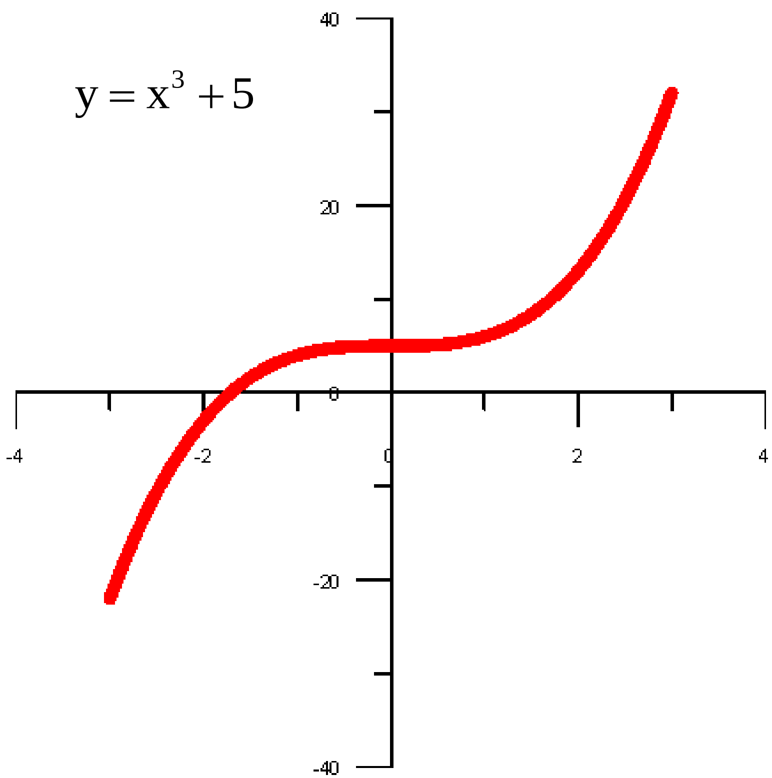


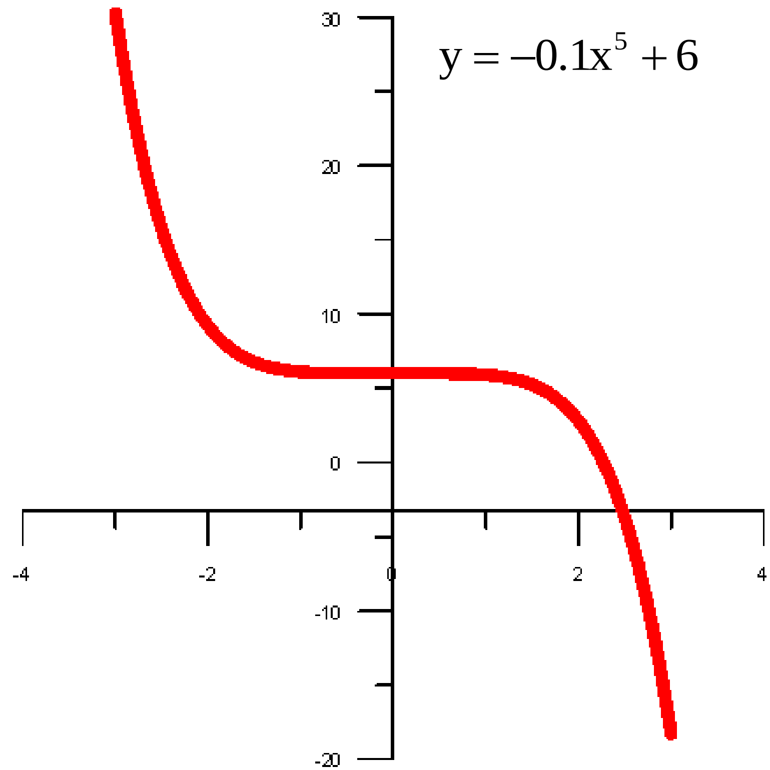
Power function

$$y = ax^b + c \qquad a, c \in R,\ b \in Z \cup (0,1)$$

2. If b is positive odd integer, the graph is “cubic style”. The “a” term controls the slope of “legs”. Sign of “a” controls orientation of graph:
   “c” controls position of the line in direction of axis y

$$D(f) = R, H(f) \ddagger R$$



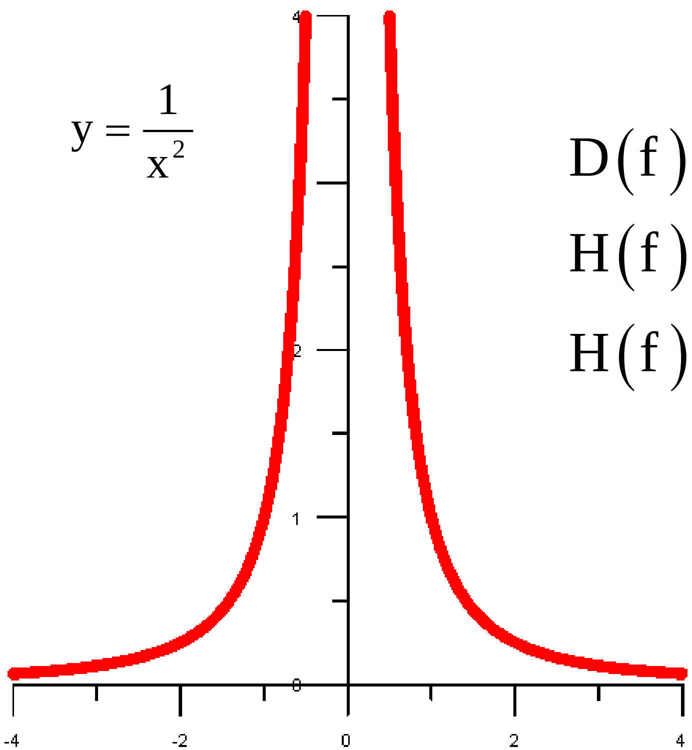


Power function

$$y = ax^{b} + c \qquad a, c \in \mathbb{R},\ b \in \mathbb{Z} \cup (0,1)$$

3. If b is negative even integer, the graph is “chimney style”

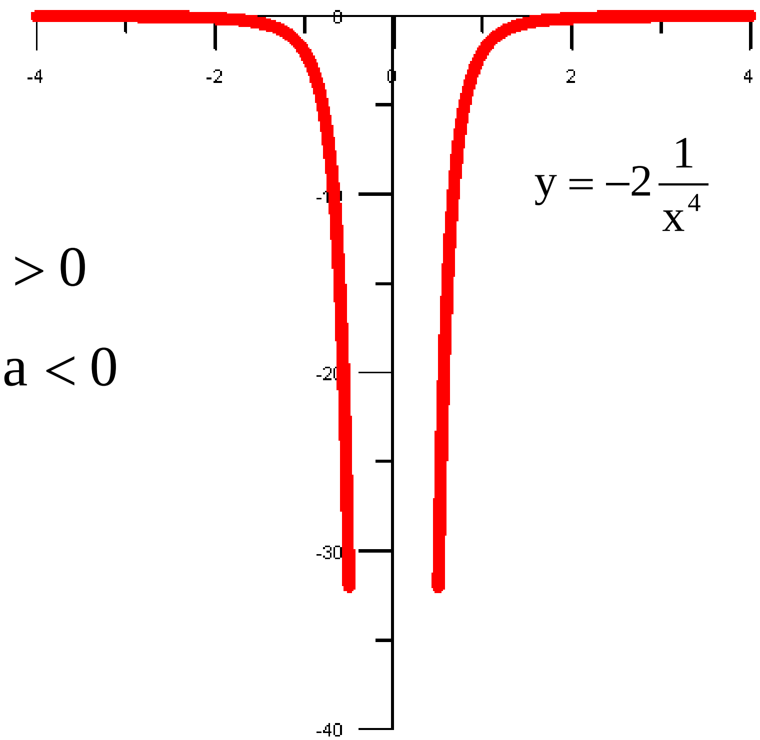
$$y = \frac{1}{x^2}$$



$$D(f) = \mathbb{R} - \{0\}$$

$$H(f) = (c, \infty),\ \text{if } a > 0$$

$$H(f) = (-\infty, c),\ \text{if } a < 0$$



$$y = -2\frac{1}{x^4}$$

Power function

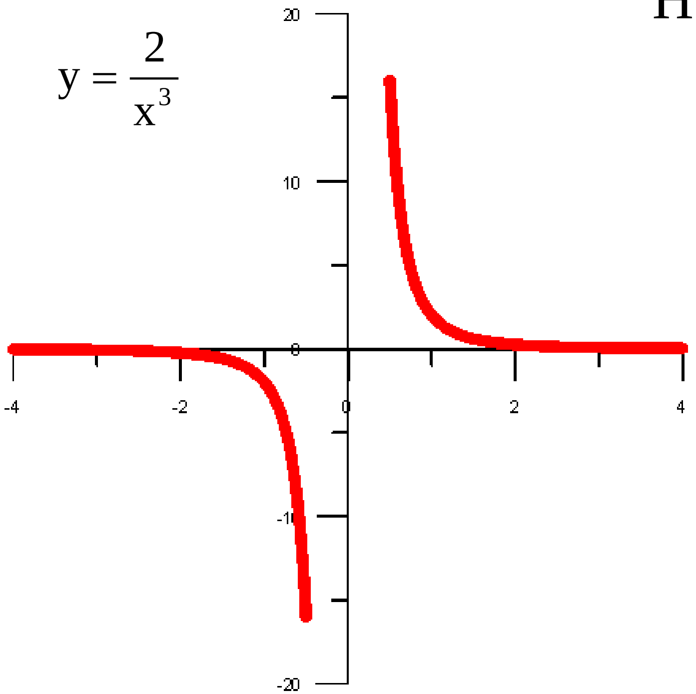
$$y = ax^{b} + c \qquad a, c \in R,\ b \in Z \cup (0,1)$$

4. If b is negative odd integer, the graph is hyperbola

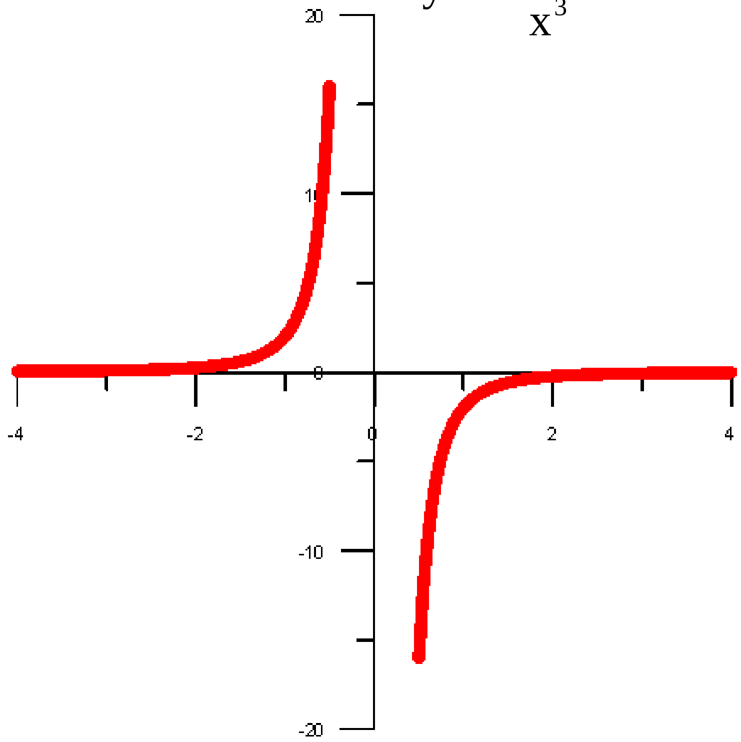
$$D(f) = R - \{0\}$$

$$H(f) = R - \{c\}$$

$$y = \frac{2}{x^3}$$



$$y = -\frac{2}{x^3}$$

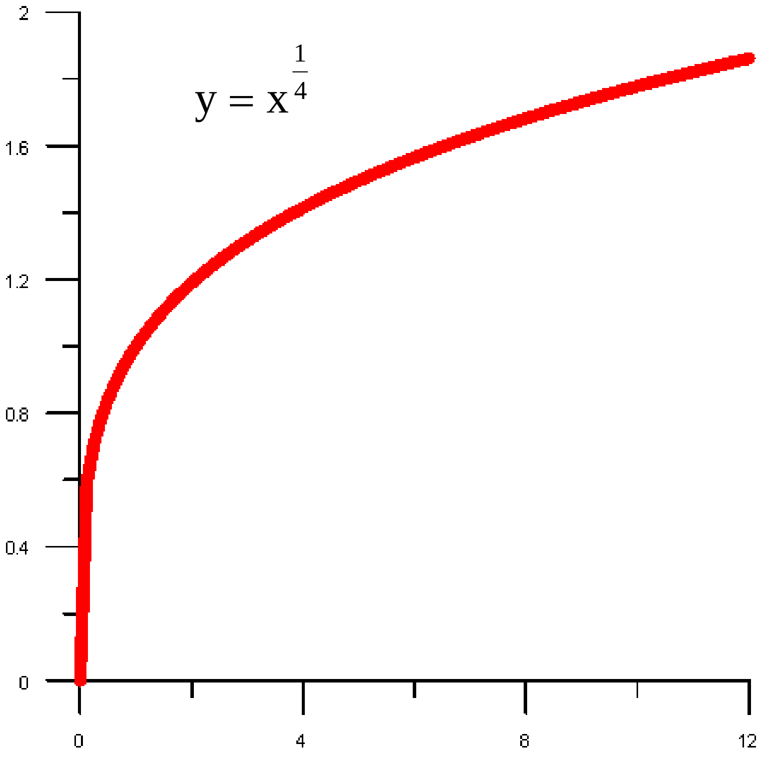


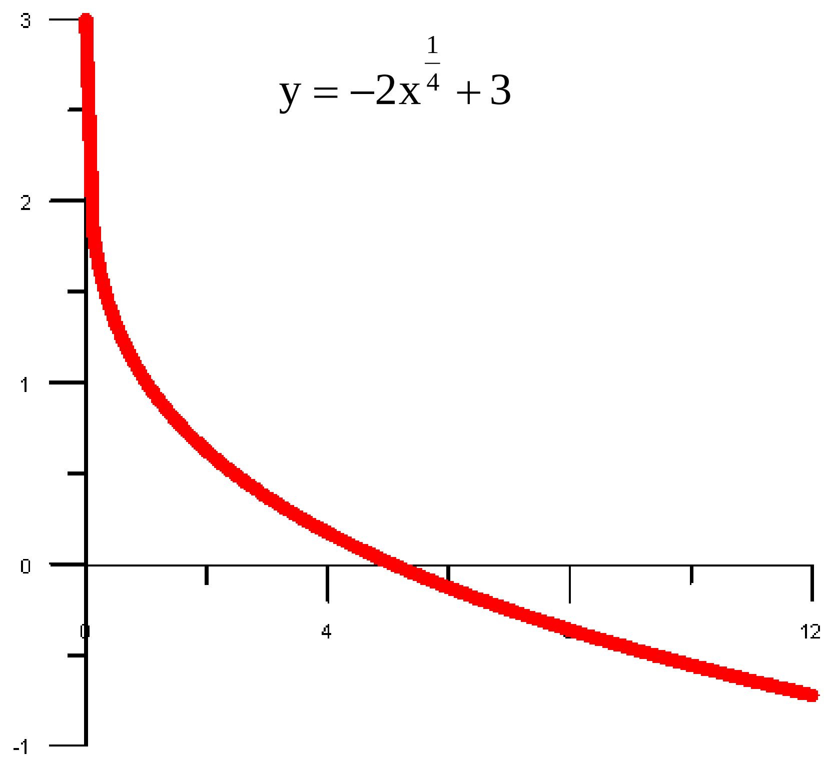
Power function

$$y = ax^b + c \qquad a, c \in \mathbb{R},\ b \in \mathbb{Z} \cup (0,1)$$

5. If $b \in (0,1)$ "root" type of graph

D(f) depends on b





Polynomial

$$y = \sum_{k=0}^{n} a_k x^k$$

The constant, linear quadratic, cubic etc. functions are special case of polynomial

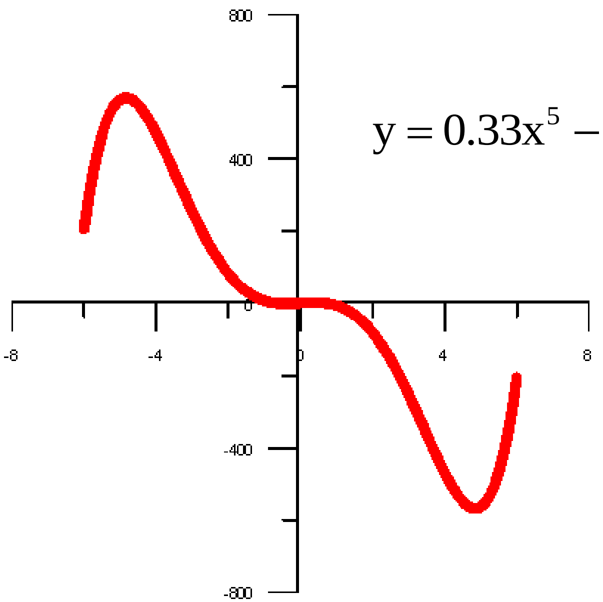
n – degree of polynomial, $n \in \mathbb{N}$

$D(f) = \mathbb{R}$

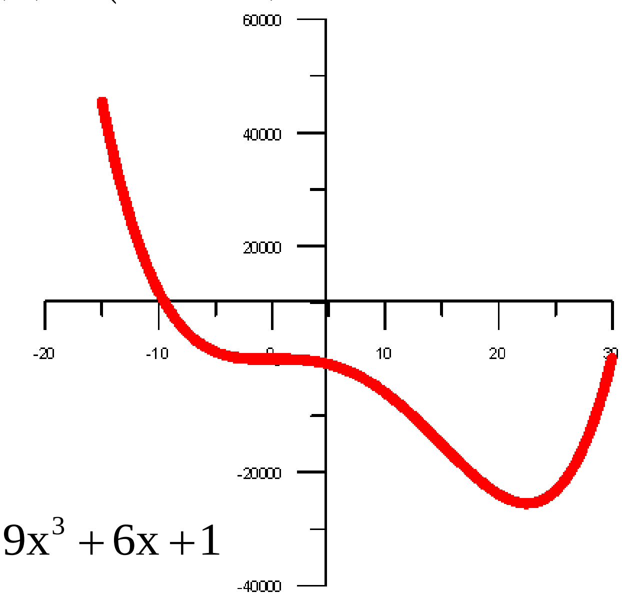
$H(f)$ depends of n, if n is odd number, H(f) = $\mathbb{R}$

if n is even there is supremum or infimum

and $H(f) = \langle \inf, \infty)$ or $H(f) = (-\infty, \sup\rangle$



$y = 0.33x^5 - 13x^3 + 6x + 1$



$y = 0.3x^4 - 9x^3 + 6x + 1$

Exponential function

$$y = ab^x + c \qquad a, c \in \mathrm{R},\ b \in \mathrm{R}^+ \qquad D(f) = \mathrm{R}$$

if a > 0

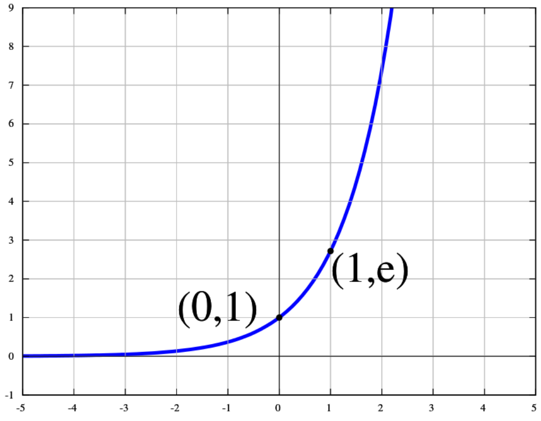
if b > 1 function is increasing,
if $b \in (0,1)$ function is decreasing

if a < 0

if b > 1 function is decreasing,
if $b \in (0,1)$ function is increasing

The exponential function occurs most frequently as: $y = e^x$

where e is Euler constant: e = 2.7182818284590…..



Logarithm function

$$y = b \cdot \log_a(x) + c \qquad b, c \in R,\ a \in R^+ - \{1\} \qquad D(f) = R^+$$

- inverse to exponential function, term “a” is called base of logarithm

Basic rule: the base involved to result is the number in logarithm:

$$y = \log_a(x) \rightarrow a^y = x$$

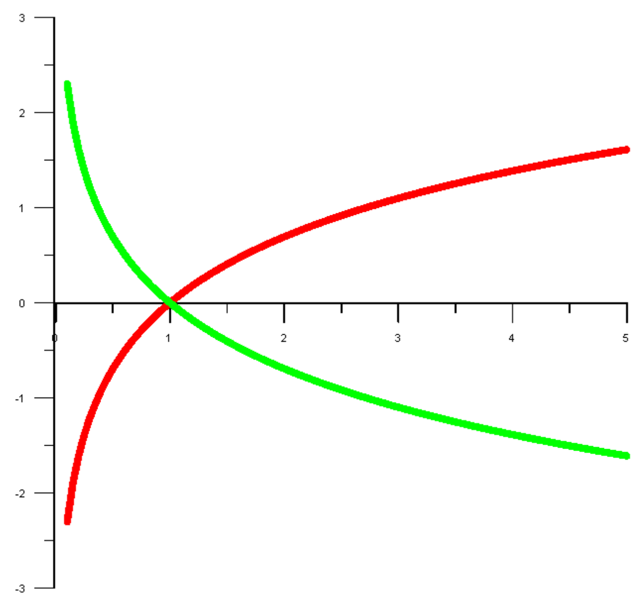
- if a = 10, we write log(x) instead of $\log_{10}(x)$
- if a = e (Euler constant), we write ln (x) instead of $\log_e(x)$

Important properties:

$$\ln a + \ln b = \ln(a \cdot b)$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$$\ln a^b = b \ln a$$



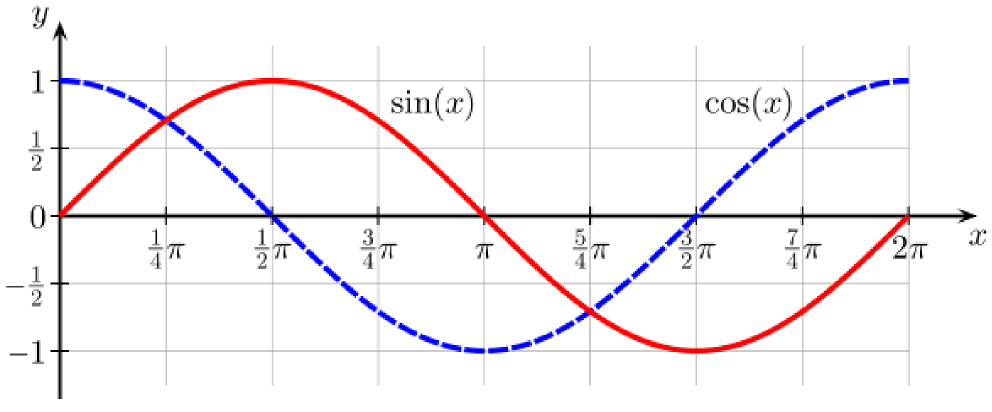
Trigonometric functions

$$y = a \cdot \sin(bx) + c,\ y = a \cdot \cos(bx) + c \qquad a, b, c \in \mathbb{R}$$

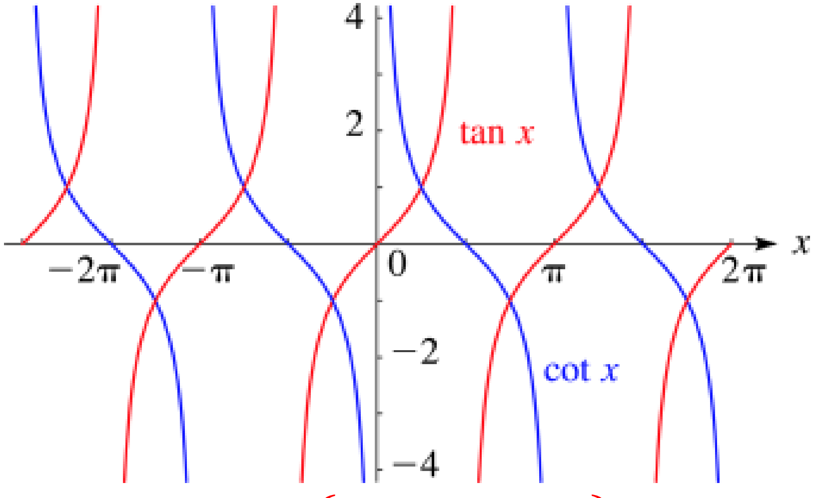
$$y = a \cdot \tan(bx) + c,\ y = a \cdot \cot(bx) + c \qquad a, b, c \in \mathbb{R}$$

Periodic functions: T = 2π /b for sin(bx) and cos(bx)

T = π /b for tan(bx) and cot (bx)



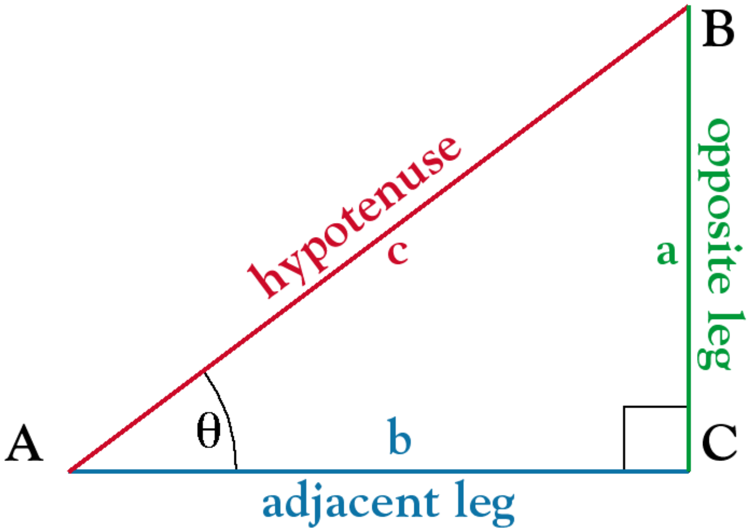
$$D(f) = \mathbb{R}$$



$$D(f) = \mathbb{R} - \left\{(2k+1)\frac{\pi}{2}\right\}, k \in \mathbb{Z}$$

$$D(f) = \mathbb{R} - \{k\pi\}, k \in \mathbb{Z}$$

Trigonometric functions



$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\csc\theta = \frac{hypotenuse}{opposite} = \frac{1}{\sin(x)}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{1}{\cos(x)}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{\sin(x)}{\cos(x)}$$

$$\cot\theta = \frac{adjacent}{opposite} = \frac{\cos(x)}{\sin(x)}$$

Trigonometric functions

Degrees	*Radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	0	1	0	–	1	–
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	–	1	–	0

Important relations

$$\sin^2 x + \cos^2 x = 1, \qquad 2\sin x \cdot \cos x = \sin 2x, \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

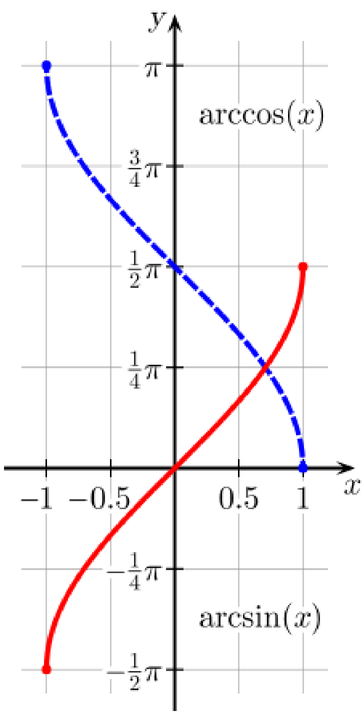
$$\cos(x + y) = \cos x \cos y \mp \sin x \sin y$$

Inverse trigonometric (cyclometric) functions

Since none of the trigonometric functions are one-to-one, they are restricted in order to have inverse functions. Therefore the ranges of the inverse functions are proper subsets of the domains of the original functions

$$y = a \cdot \arcsin(bx) + c,\ y = a \cdot \arccos(bx) + c$$

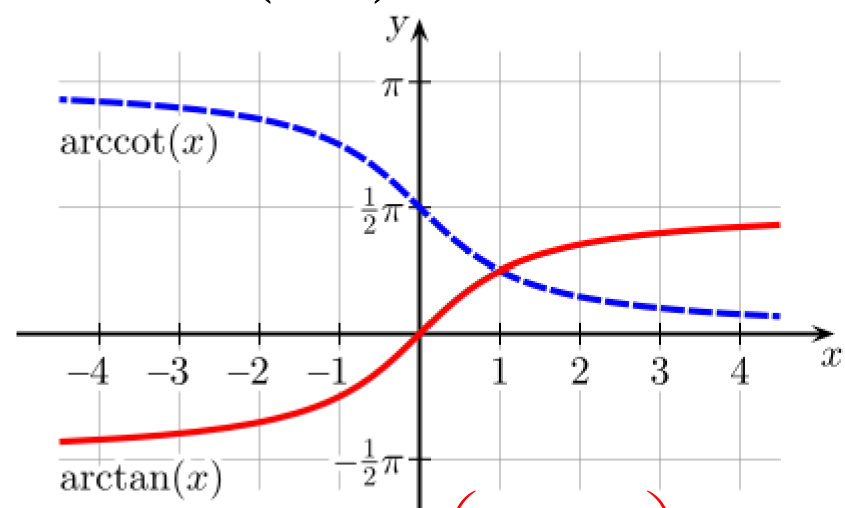
$$y = a \cdot \arctan(bx) + c,\ y = a \cdot \operatorname{arccot}(bx) + c$$



$$D(f) = \langle -1, 1 \rangle$$

$$H(f) = \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$$

$$H(f) = \langle 0, \pi \rangle$$



$D(f) = R$, $H(f) = \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

$$H(f) = (0, \pi)$$

Hyperbolic functions

$$y = a \cdot \sinh(bx) + c, \; y = a \cdot \cosh(bx) + c$$

$$y = a \cdot \tanh(bx) + c, \; y = a \cdot \coth(bx) + c$$

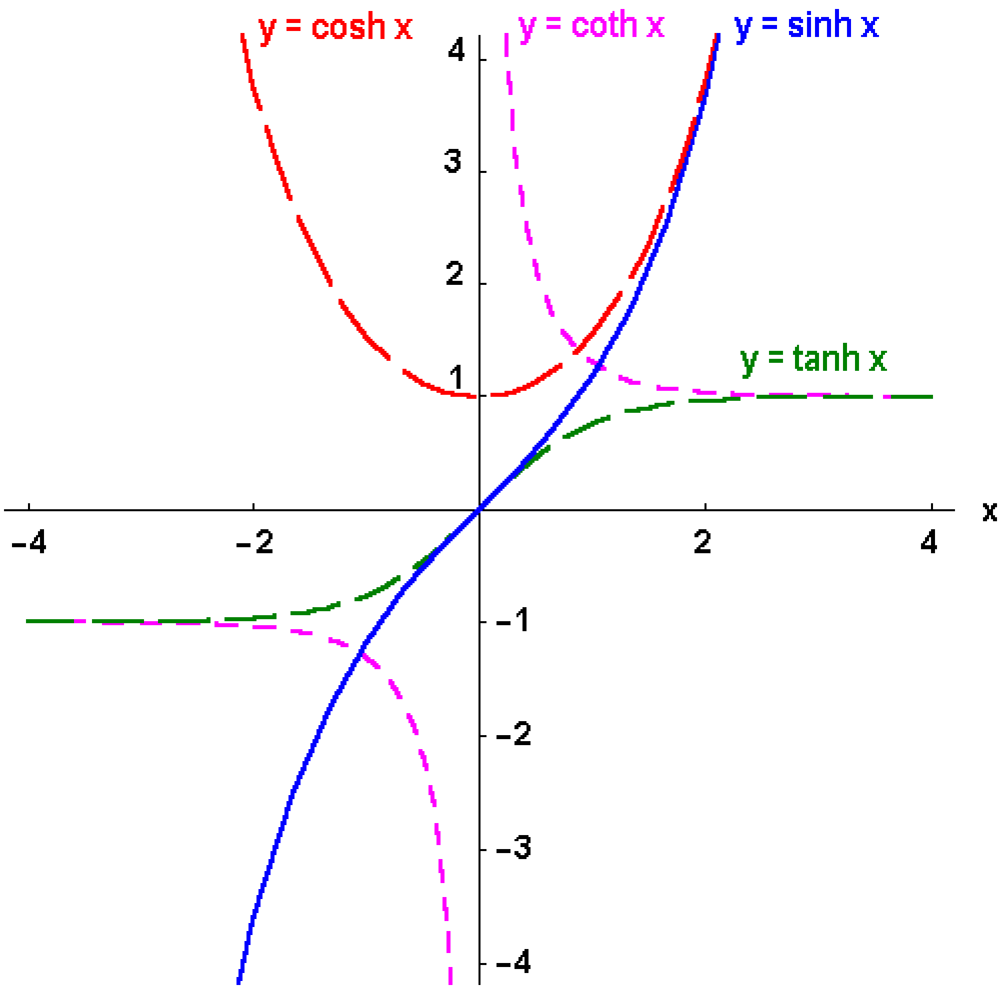
Combinations of exponential functions:

$$y = \sinh(x) = \frac{e^{x} - e^{-x}}{2}, \; y = \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$y = \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}},$$

$$y = \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

Hyperbolic functions



$$D(f) = \mathbb{R}$$

Inverse hyperbolic functions

$$y = a \cdot \arg\sinh(bx) + c,\ y = a \cdot \arg\cosh(bx) + c$$

$$y = a \cdot \arg\tanh(bx) + c,\ y = a \cdot \arg\coth(bx) + c$$

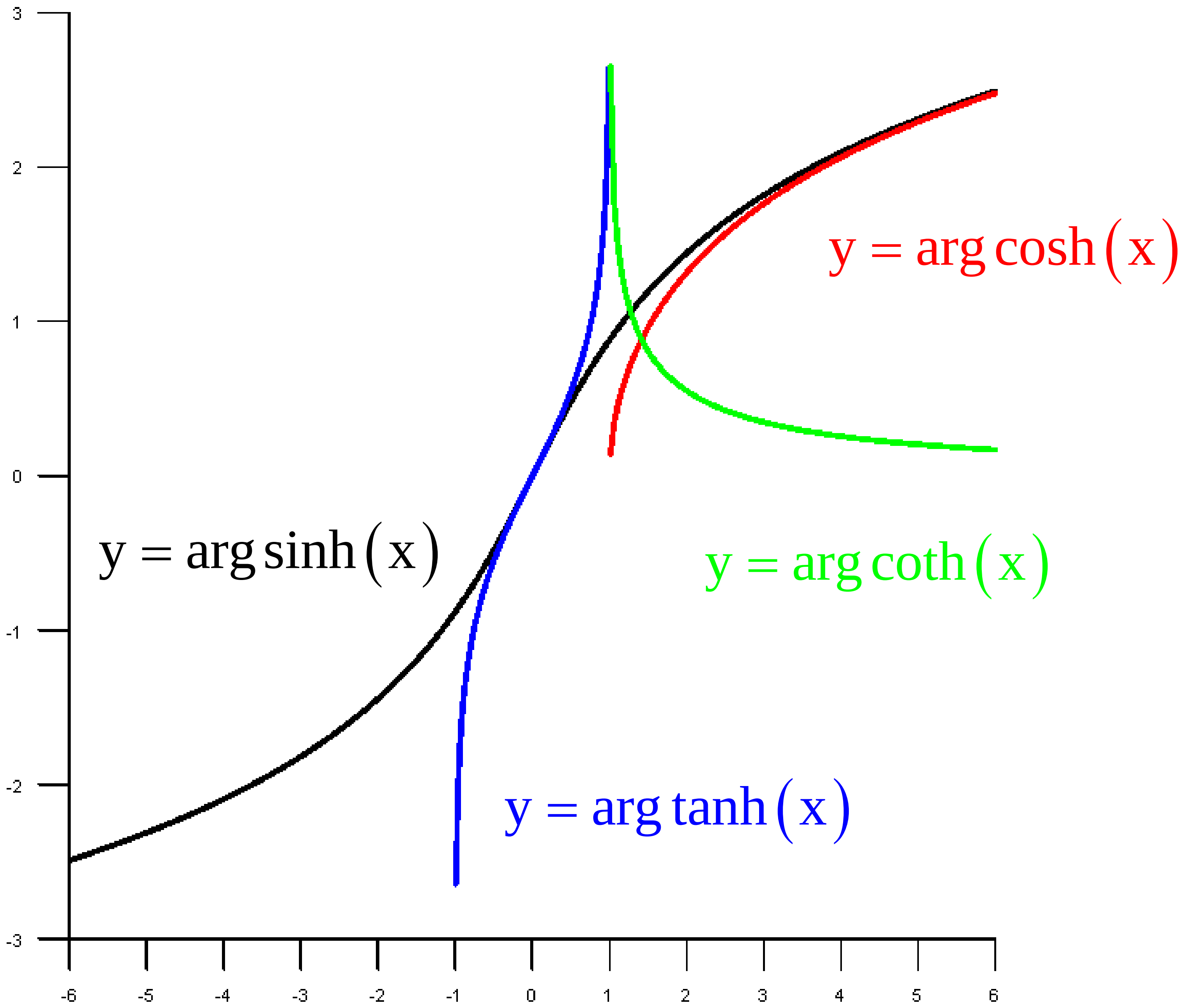
Special type of logarithm:

$$y = \arg\sinh(x) = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad D(f) = \mathrm{R}$$

$$y = \arg\cosh(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad D(f) = \langle 1, \infty)$$

$$y = \arg\tanh(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad D(f) = (-1, 1)$$

$$y = \arg\coth(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad D(f) = \mathrm{R} - \langle -1, 1\rangle$$



y = arg cosh (x)
y = arg sinh (x)
y = arg coth (x)
y = arg tanh (x)
3
2
1
0
-1
-2
-3
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6