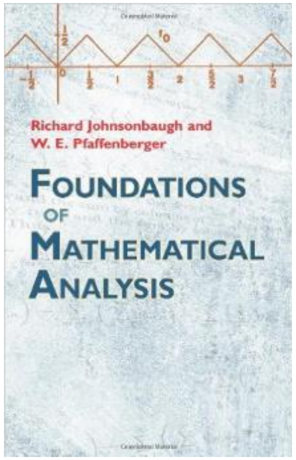
Lecture 1: Basic terms and rules in mathematics

Basic info:

1. Organisation of the term (semester) and evaluation of the subject, all lectures on the website www.kaeg.sk, with each lecture also a small vocabulary will be given (in the end).
2. Evaluation of the subject – 100% final examination (few definitions and solution of exercises)
3. Basic literature (internet sources: wiki, Wolfram MathWorld, …).

Lecture 1: Basic terms and rules in mathematics

Basic literature:



"official sources":

Johnsonbaugh R., Pfaffenberger W.E., 2010:
Foundations of mathematical analysis. Dover.

Apostol T.M., 1974:
Mathematical Analysis, 2nd edition, Addison-Wesely.

Dettman J.W. 1968:
Introduction to linear algebra and differential equations. Dover.

But presentations from lectures should be enough (you can find them at: www.kaeg.sk).

We also advise a Slovak text-book for terminology:

Gombárik P., 2012: Anglický jazyk pre študentov FMFI UK, Matematika, UK v Bratislave

Lecture 1: Basic terms and rules in mathematics

Content:

- introduction (why so much Math in BCH?),
- repetition of logical operators,
- numbers, number systems,
- sets, basic operations with sets,
- matrices, operations with matrices,
- determinants.

Topic: Basic terms in mathematics

„Mathematics is the scientific language of natural sciences.“

Topic: Basic terms in mathematics

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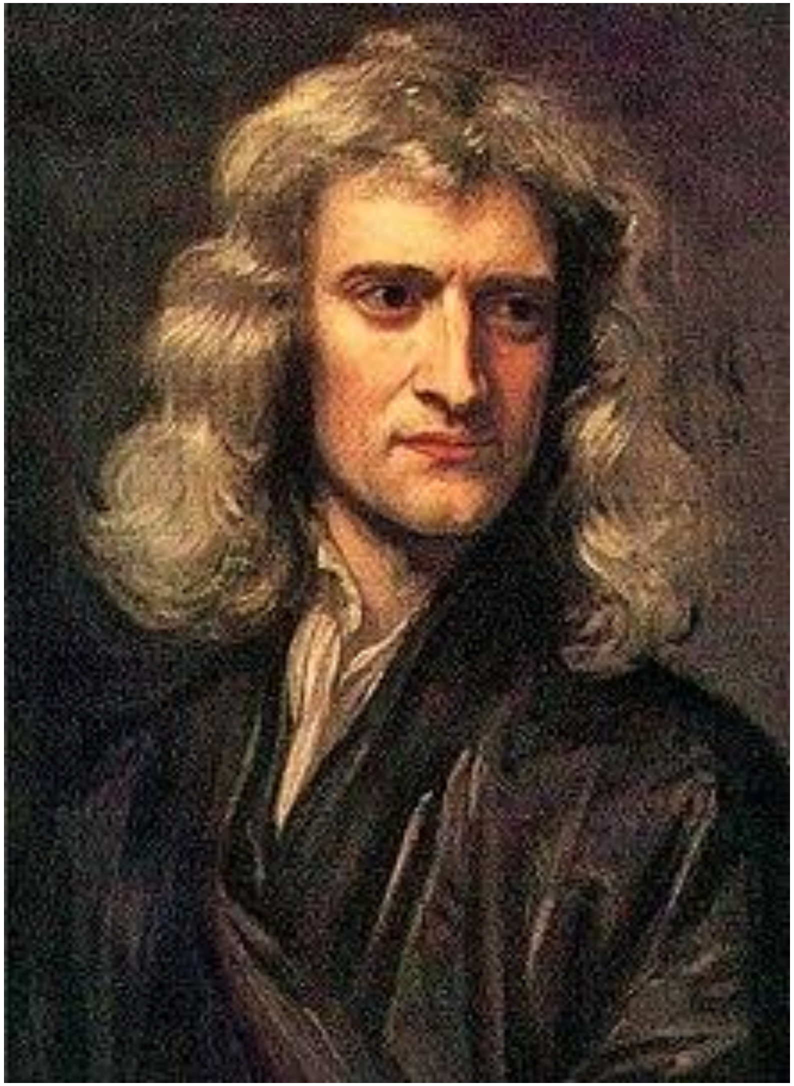

Isaac Newton (1643 - 1727)

Topic: Basic terms in mathematics

Mathematics is a science of structure, order, and relation that has evolved from elemental practices of <u>counting</u>, <u>measuring</u>, and <u>describing</u> the objects.

It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter (Encyclopaedia Britannica).

In chemistry and biology we utilise mathematical methods not only for counting and measuring (e.g. with statistics), but also in other aspects – in objects description and abstraction.

Topic: Basic terms in mathematics

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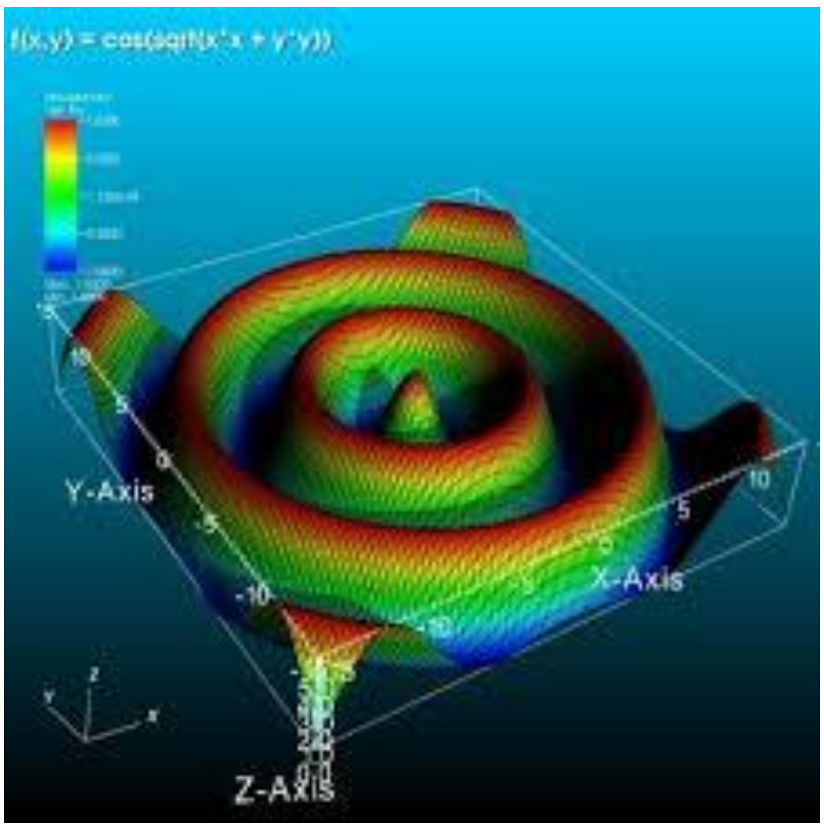
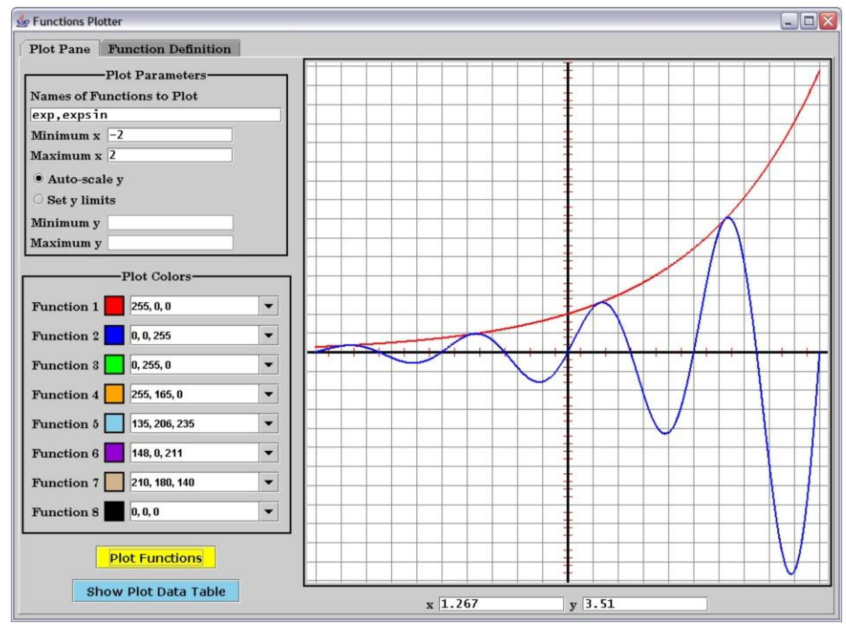


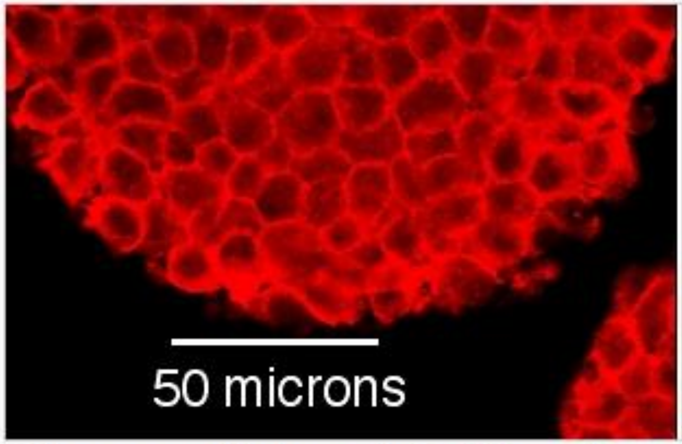


speaking about measures – some conversions:

Prefix	Symbol	1000^m	10^n	Decimal	Short scale	Long scale	Since[n 1]
yotta	Y	1000^{8}	10^{24}	1 000 000 000 000 000 000 000 000	Septillion	Quadrillion	1991
zetta	Z	1000^{7}	10^{21}	1 000 000 000 000 000 000 000	Sextillion	Trilliard	1991
exa	E	1000^{6}	10^{18}	1 000 000 000 000 000 000	Quintillion	Trillion	1975
peta	P	1000^{5}	10^{15}	1 000 000 000 000 000	Quadrillion	Billiard	1975
tera	T	1000^{4}	10^{12}	1 000 000 000 000	Trillion	Billion	1960
giga	G	1000^{3}	10^{9}	1 000 000 000	Billion	Milliard	1960
mega	M	1000^{2}	10^{6}	1 000 000	Million		1960
kilo	k	1000^{1}	10^{3}	1 000	Thousand		1795
hecto	h	$1000^{2/3}$	10^{2}	100	Hundred		1795
deca	da	$1000^{1/3}$	10^{1}	10	Ten		1795
		1000^{0}	10^{0}	1	One		–
deci	d	$1000^{-1/3}$	10^{-1}	0.1	Tenth		1795
centi	c	$1000^{-2/3}$	10^{-2}	0.01	Hundredth		1795
milli	m	1000^{-1}	10^{-3}	0.001	Thousandth		1795
micro	µ	1000^{-2}	10^{-6}	0.000 001	Millionth		1960
nano	n	1000^{-3}	10^{-9}	0.000 000 001	Billionth	Milliardth	1960
pico	p	1000^{-4}	10^{-12}	0.000 000 000 001	Trillionth	Billionth	1960
femto	f	1000^{-5}	10^{-15}	0.000 000 000 000 001	Quadrillionth	Billiardth	1964
atto	a	1000^{-6}	10^{-18}	0.000 000 000 000 000 001	Quintillionth	Trillionth	1964
zepto	z	1000^{-7}	10^{-21}	0.000 000 000 000 000 000 001	Sextillionth	Trilliardth	1991
yocto	y	1000^{-8}	10^{-24}	0.000 000 000 000 000 000 000 001	Septillionth	Quadrillionth	1991

1. ^ The metric system was introduced in 1795 with six prefixes. The other dates relate to recognition by a resolution of the CGPM.

www.bustatech.com



Cells are so small that even a cluster of these cells from a mouse only measures 50 microns

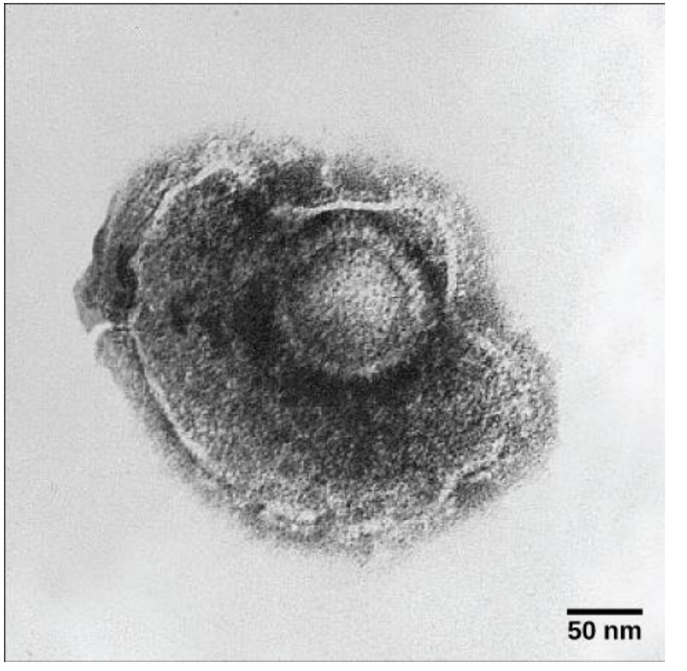
micron is a non-SI name for micrometre ($\mu m = 10^{-6}$ m)

a next useful and used term:
ppm – parts per million
(used beside percent, permile)

1 % = 10 000 ppm
1 ‰ = 1 000 ppm
ppb – parts per billion

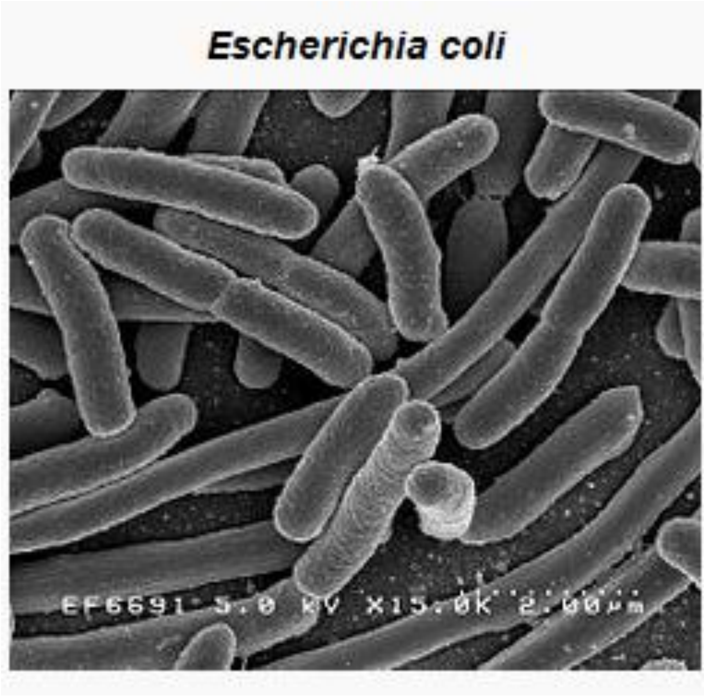
some examples of sizes from biology (1/2):

0.1 nm (nanometer) diameter of a hydrogen atom
0.8 nm Amino Acid
2 nm Diameter of a DNA Alpha helix
4 nm Globular Protein
6 nm microfilaments
7 nm thickness cell membranes
20 nm Ribosome
25 nm Microtubule
30 nm Small virus (Picornaviruses)
30 nm Rhinoviruses
50 nm Nuclear pore
100 nm HIV
120 nm Large virus (Orthomyxoviruses, includes influenza virus)
150-250 nm Very large virus (Rhabdoviruses, Paramyxoviruses)
150-250 nm small bacteria such as Mycoplasma
200 nm Centriole
200 nm (200 to 500 nm) Lysosomes
200 nm (200 to 500 nm) Peroxisomes
800 nm giant virus Mimivirus



scan from an electron microscope (a virus)

some examples of sizes from biology (2/2):



1 µm (micrometer)

(1 - 10 µm) the general sizes for Prokaryotes

1 µm Diameter of human nerve cell process

2 µm E.coli - a bacterium

3 µm Mitochondrion

5 µm length of chloroplast

6 µm (3 - 10 micrometers) the Nucleus

9 µm Human red blood cell

10 µm

(10 - 30 µm) Most Eukaryotic animal cells

(10 - 100 µm) Most Eukaryotic plant cells

90 µm small Amoeba

100 µm Human Egg

up to 160 µm Megakaryocyte

up to 500 µm giant bacterium Thiomargarita

up to 800 µm large Amoeba

1 mm (1 millimeter, 1/10th cm)

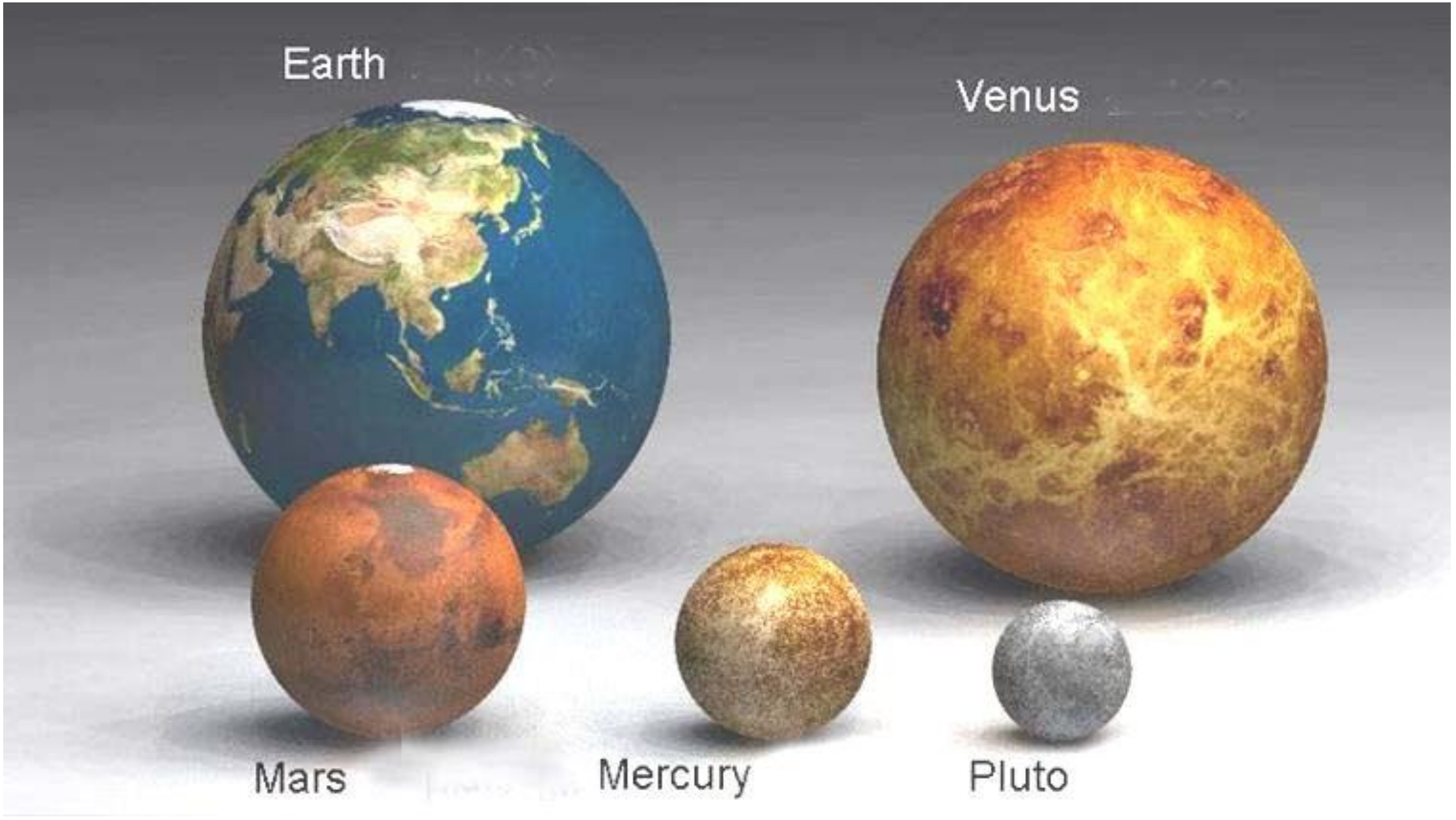
1 mm Diameter of the squid giant nerve cell

up to 40mm Diameter of giant amoeba Gromia Sphaerica

120 mm Diameter of an ostrich egg (a dinosaur egg was much larger)

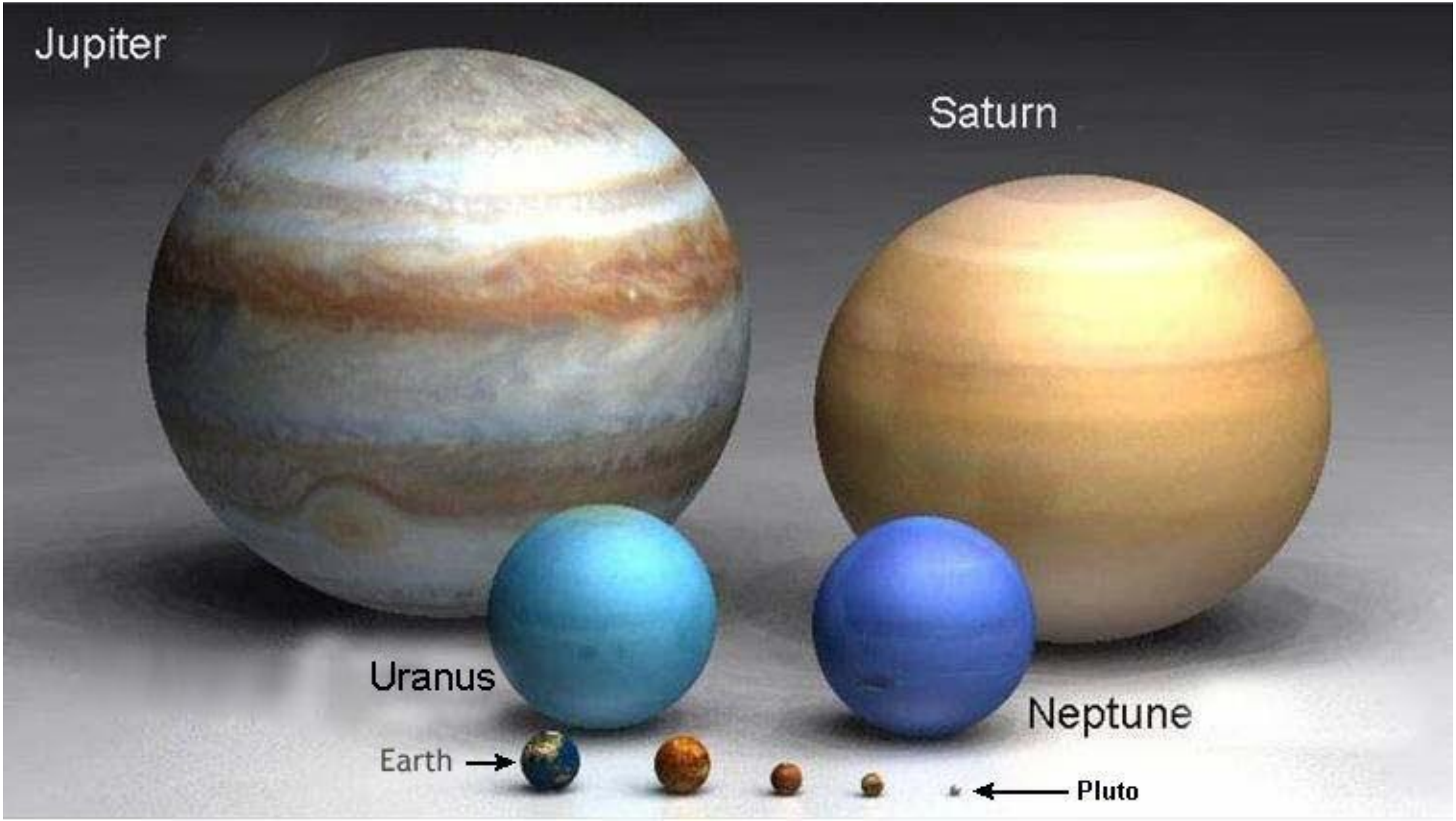
3 meters Length of a nerve cell of giraffe's neck

some larger dimensions...



diameter of the Earth: 12756.2 kilometers ≈ $12.8\ 10^6$ meters

some larger dimensions...



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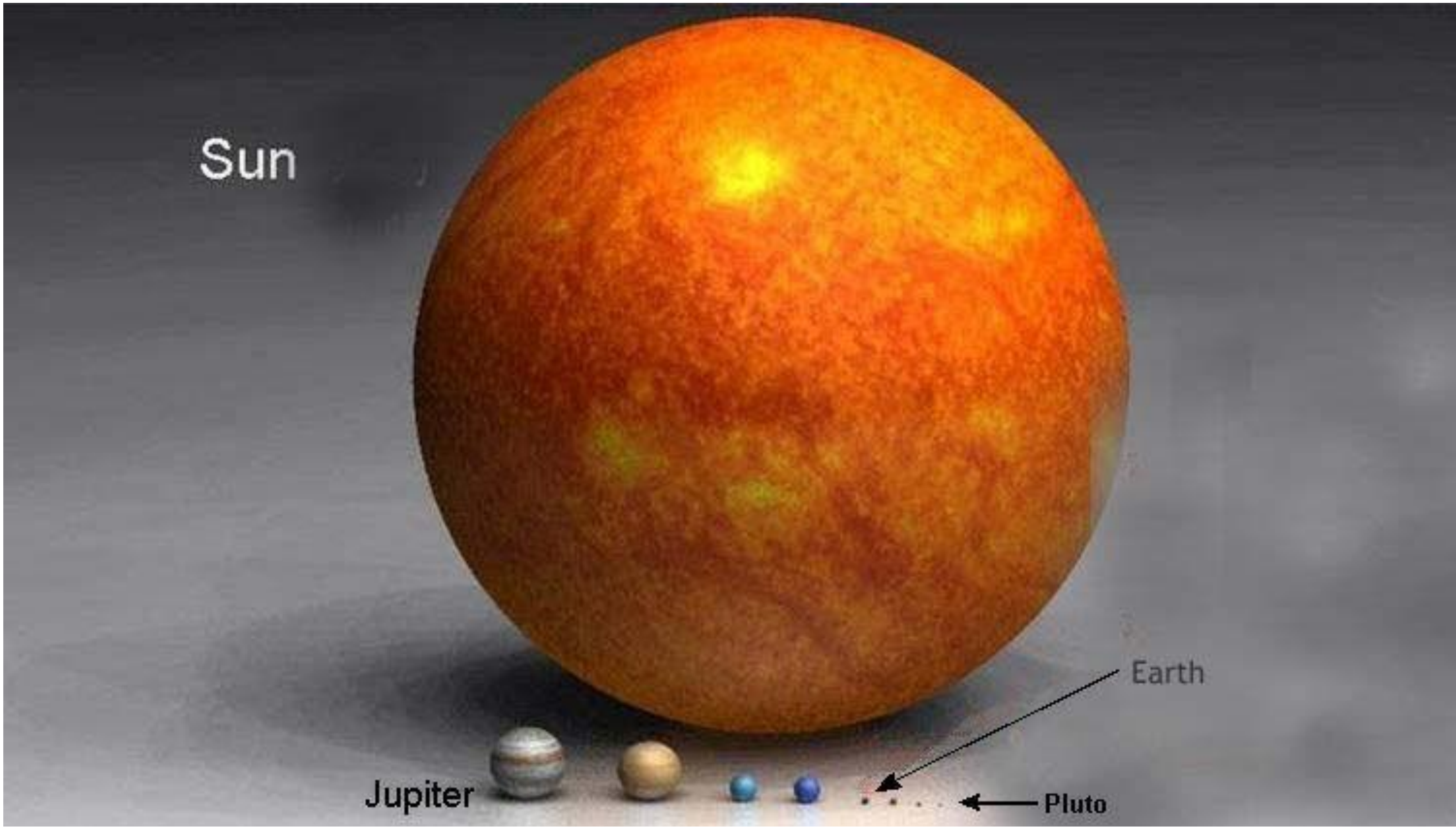
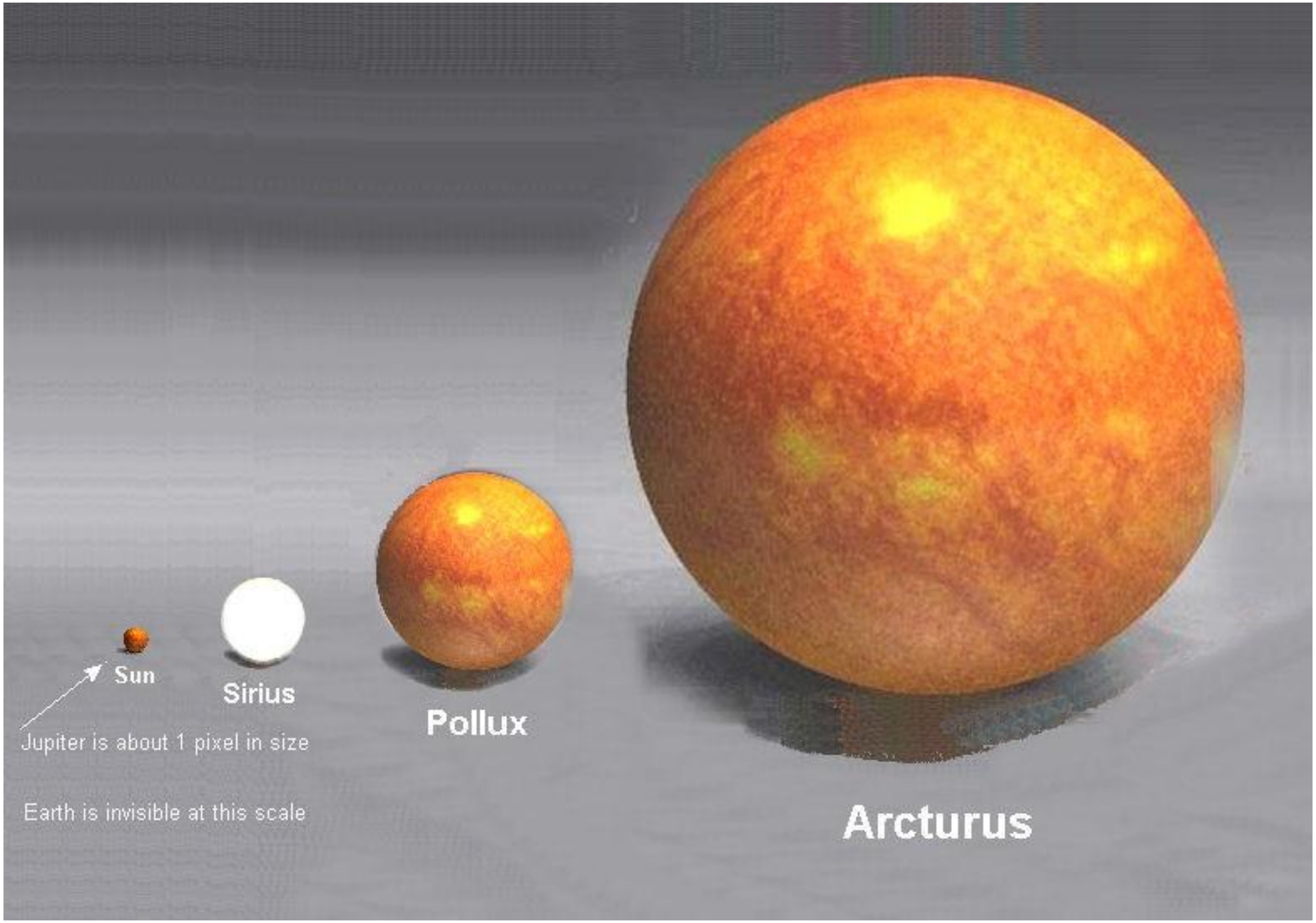


diameter of the Sun: 1391000 kilometers = 1.391 10^9 meters

some larger dimensions... (outside solar system)



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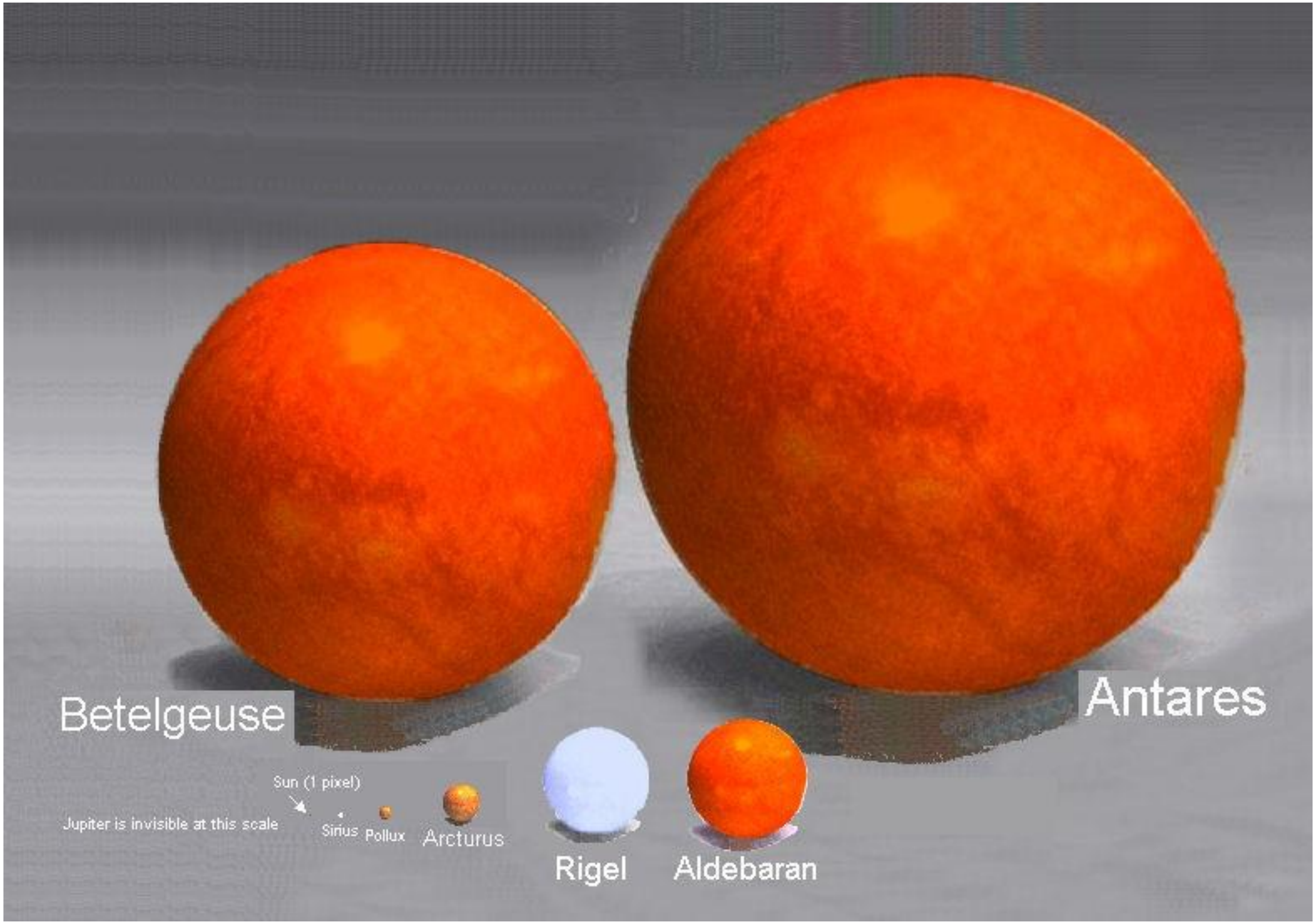


diameter of the Sun: 1391000 kilometers = 1.391 10^9 meters

Topic: Basic terms in mathematics

There exists many branches is mathematics:

- <u>algebra</u>, arithmetic, <u>analysis</u>, <u>statistics</u>, numerical mathematics, optimisation,...

Topic: Basic terms in mathematics

Basic objects in mathematics:

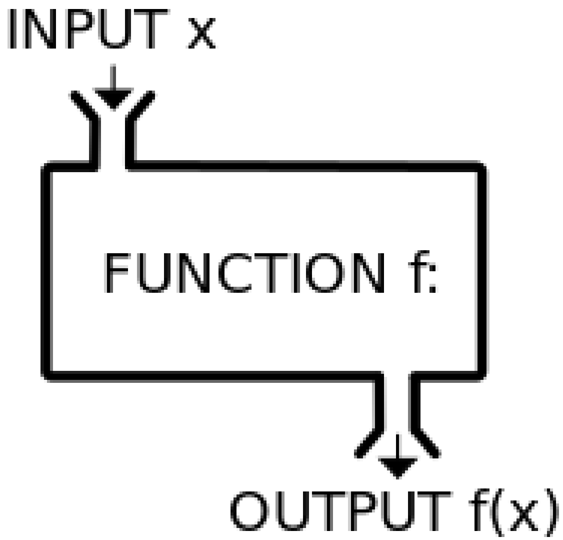
- numbers, functions, functionals.

A **number** is a mathematical object used to count, measure and label. Numbers can be classified into sets, called number systems, such as the natural numbers and the real numbers (more details will come in a moment).

$$1, 2, 3, 4, \cdots$$

To generalize the work with numbers, we often use instead of direct numbers variables.

A **variable** is an alphabetic character representing a number (e.g. x, a).

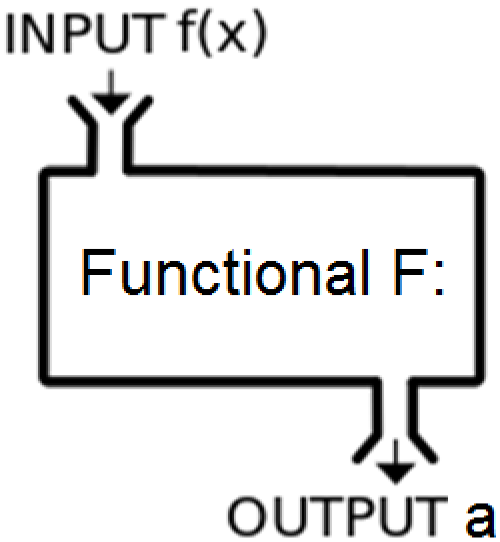


A **function** is a relation between a set of inputs and a set of permissible outputs with the propert that each input is related to exactly one output (we will come to it in more detail later on).

Topic: Basic terms in mathematics

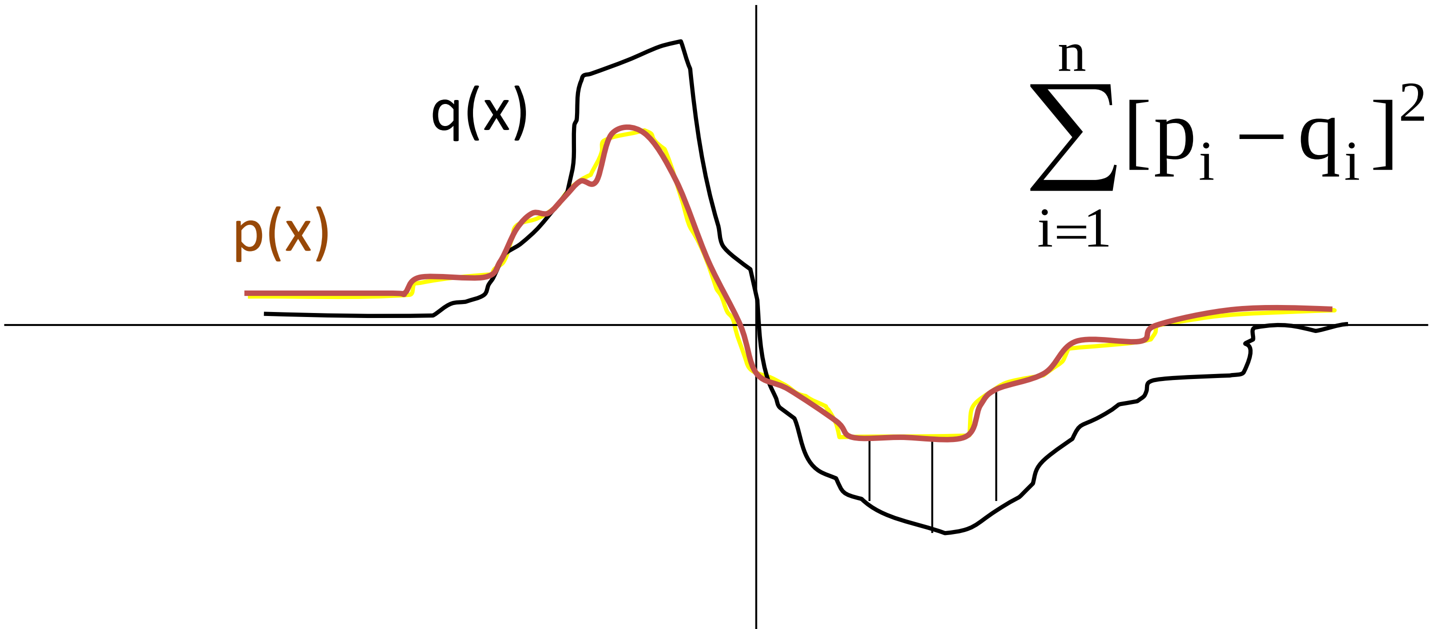
Basic objects in mathematics:

- numbers, functions, functionals.



A **functional** is a mathematical object (operator), which has in the input a function (even more functions) and in the output a number (variable).

Example: so called Least Squares (LSQ) functional



$$\sum_{i=1}^{n} [p_i - q_i]^2 = a$$

Comment: Different symbolism in math

Mathematics use very **strict rules** in the formalism.

Even a difference in a used style of writing (e.g. normal or **bold** or *italics*) can express important differences in the used meaning (e.g. between matrices and usual variables or in physics we distinguish between scalars and vectors).

Of course that there exists differences in the variety of text-books, but some rules are valid in general.

We will try to hold these rules also in this subjects, but errors can happen, so please let us know when you will find some of them in the lectures or exercises. Thanks in anticipation.

Additional comment: In this kind of lectures we will often mix traditional math with numerical math (show examples for solutions with also digitized functions, not only continuous ones).

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Topic: Basic terms in mathematics

Logical operators in mathematics (Boolean algebra):

In mathematics and mathematical logic, Boolean algebra is the branch of algebra in which the values of the variables are the truth values: true and false, usually denoted 1 and 0 respectively.

Instead of elementary algebra,

the **main operations of Boolean algebra** are:

- the conjunction and (AND), denoted $\wedge$

  (satisfies $x \wedge y = 1$ if $x = y = 1$ and $x \wedge y = 0$ otherwise),

- the disjunction or (OR), denoted $\vee$

  (satisfies $x \vee y = 0$ if $x = y = 0$ and $x \vee y = 1$ otherwise),

- the negation not (NOT), denoted $\neg$

  (satisfies $\neg x = 0$ if $x = 1$ and $\neg x = 1$ if $x = 0$).

Topic: Basic terms in mathematics

Logical operators in mathematics (Boolean algebra):

There exists a next group of derived logical operators: NAND, NOR, XOR:

- NAND and NOR are negations of AND and OR

- XOR comes from Exclusive OR operation; that is a true output if one, and only one, of the inputs to the operation is true.

INPUT		OUTPUT
A	B	A NOR B
0	0	1
0	1	0
1	0	0
1	1	0

INPUT		OUTPUT
A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

Topic: Basic terms in mathematics

Statement:

In general – it is a message that is stated or declared.

Statement is intensively connected with the term proposition:

“Propositions are the sharable objects of attitudes and the primary **bearers of truth and falsity**.”

In this sense, propositions are **"statements" that are truth-bearers**. (wiki)

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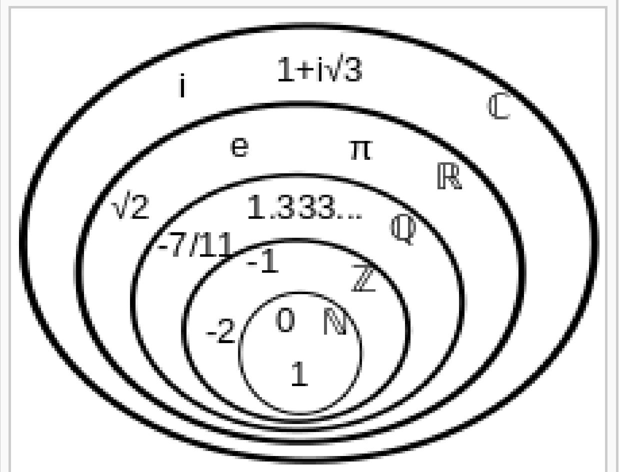
Basic objects in mathematics: **numbers, sets**

Different types of numbers have many different uses.
Numbers can be classified into sets, called **number systems**.



Subsets of the complex numbers.

Main number systems

$\mathbb{N}$	**Natural**	0, 1, 2, 3, 4, ... **or** 1, 2, 3, 4, ...
$\mathbb{Z}$	**Integer**	..., −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, ...
$\mathbb{Q}$	**Rational**	$\frac{a}{b}$ where *a* and *b* are integers and *b* is not 0
$\mathbb{R}$	**Real**	The limit of a convergent sequence of rational numbers
$\mathbb{C}$	**Complex**	*a* + *bi* where *a* and *b* are real numbers and *i* is the square root of −1

$1.5 = \frac{3}{2}$ Ratio

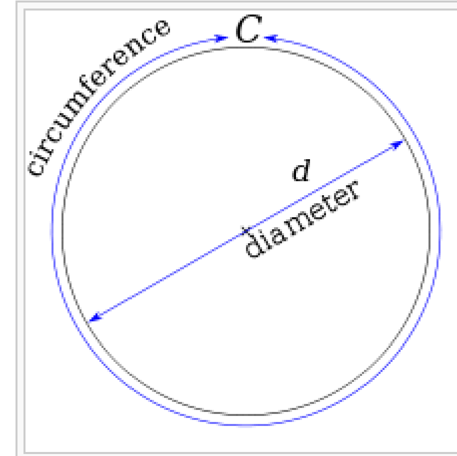
Rational

$\pi = 3.14159... = \frac{?}{?}$ (No Ratio)

Irrational

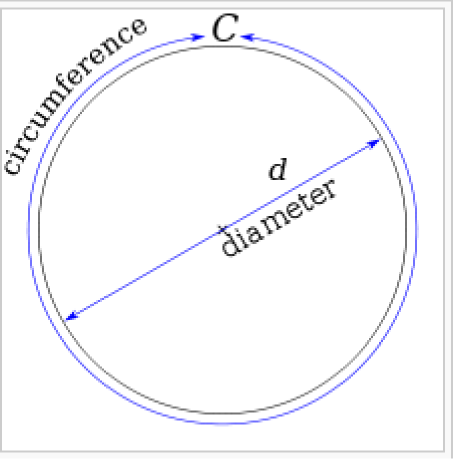
Basic objects in mathematics: **numbers, sets**

π



The number π is a mathematical constant, the ratio of a circle's circumference to its diameter, commonly approximated as 3.14159.

Basic objects in mathematics: **numbers, sets**



π

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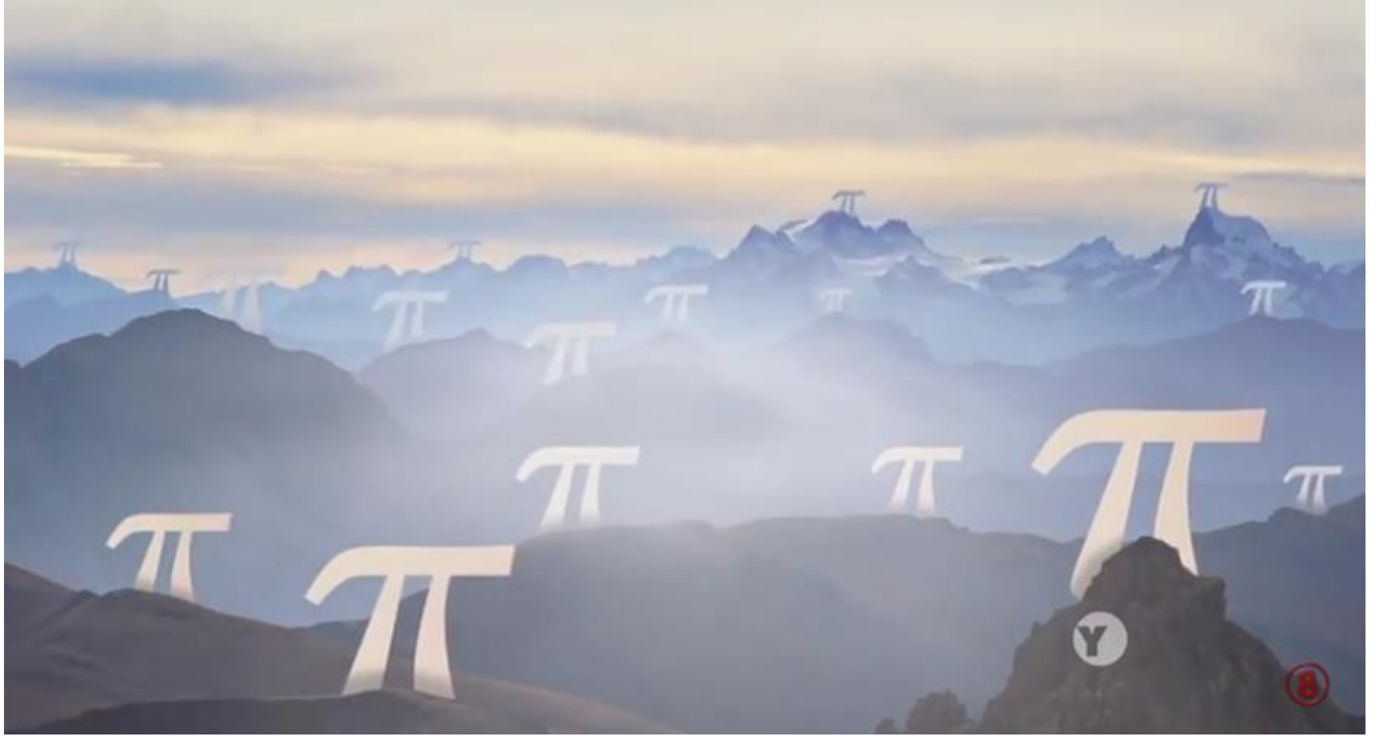
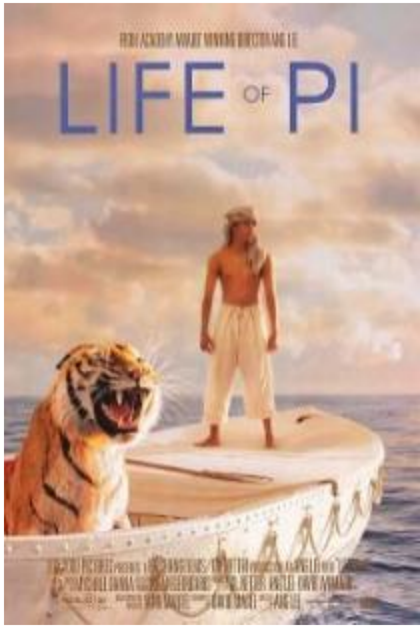




3.14159265358979323846264338327950288419716939937510...

Basic objects in mathematics: **numbers, sets**

Definition: A **set** is a collection of numbers (in general objects).

e.g. {1, 2, 3} - a set whose objects are positive numbers 1, 2, 3.

If a set consists of a finite number of objects, we may denote the set by listing them. If a set consists of an infinite many objects, we denote the set: a) by naming a property common to all objects of the set

e.g. $\{x \mid x \text{ is a positive number}\}$

(comment: the bar "|" should be read "such that")

b) in some conventional way (**N**, **Z**, **Q**, **R**, $\mathbf{R}^+$...).

If p is an object in a set **A**, we write: $p \in \mathbf{A}$
(and we say "p is an element of **A**").
If p is not an element of the set **A**, we write: $p \notin \mathbf{A}$.
E.g. If **Z** denotes the set of integers, then : $1 \in \mathbf{Z}$, but $\frac{1}{2} \notin \mathbf{Z}$.

Basic objects in mathematics: **sets**

Definitions: If **A** and **B** are sets, the union of **A** and **B** is the set

$$\mathbf{A}\cup\mathbf{B} = \{x \mid x\in\mathbf{A} \text{ or } x\in\mathbf{B}\}.$$

The intersection of **A** and **B** is the set

$$\mathbf{A}\cap\mathbf{B} = \{x \mid x\in\mathbf{A} \text{ and } x\in\mathbf{B}\}.$$

In the case, when we have several (n) sets $\mathbf{A}_i$, we can realize the union/intersection of all of them at once:

$$\mathbf{D} = \bigcup_{i=1}^{n} \mathbf{A}_i \qquad \mathbf{E} = \bigcap_{i=1}^{n} \mathbf{A}_i$$

If **A** and **B** are sets, the difference of **A** and **B** is the set

$$\mathbf{A}\setminus\mathbf{B} = \{x \mid x\in\mathbf{A} \text{ and } x\notin\mathbf{B}\}.$$

If every member of set **A** is also a member of set **B**, then **A** is said to be a subset of **B**, written $\mathbf{A}\subseteq\mathbf{B}$ (pronounced: „A is contained in B“).

Empty set is the set with no element, it is denoted $\varnothing$.

Basic objects in mathematics: **sets**

<u>Example:</u>

Let
A = {1, 2, 3, 4},
B = {1, 3, 5, 7, 9}.

Compute the following sets:

a) $\mathbf{A} \cup \mathbf{B}$
b) $\mathbf{A} \cap \mathbf{B}$
c) $\mathbf{A} \setminus \mathbf{B}$
d) find some subset of **B**.

Basic objects in mathematics: **sets**

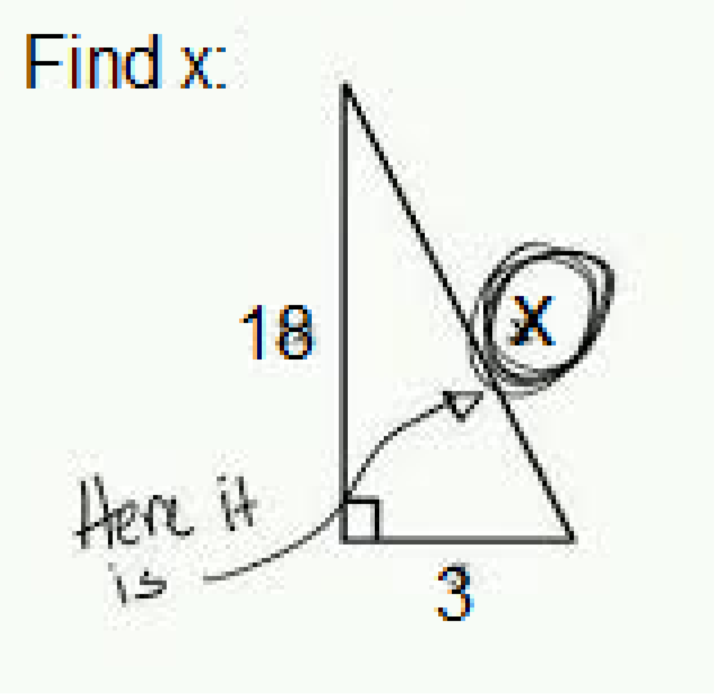
<u>Example:</u>

Let
A = {1, 2, 3, 4},
B = {1, 3, 5, 7, 9}.

Compute the following sets:

a) $\mathbf{A \cup B}$
b) $\mathbf{A \cap B}$
c) $\mathbf{A \setminus B}$
d) find some subset of **B**.

a joke ;-)



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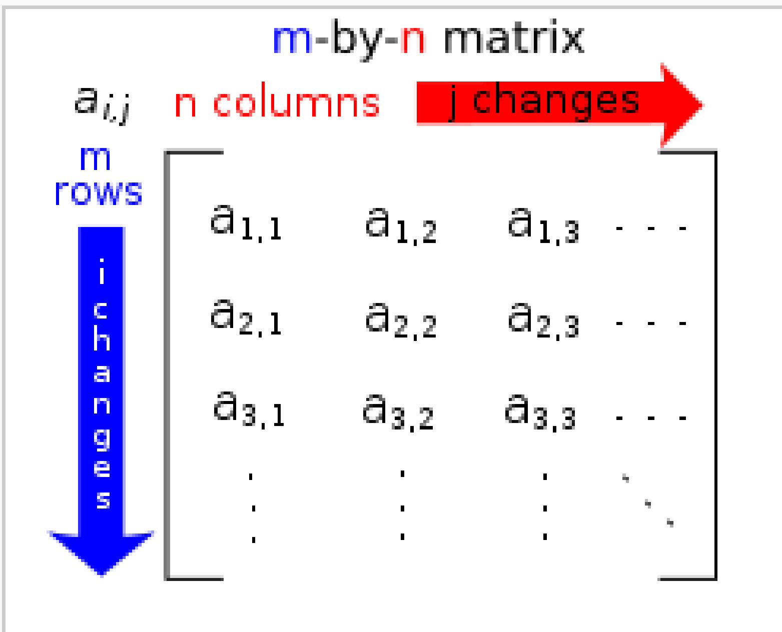
Basic objects in mathematics: **matrix (matrices)**

Definition: A **matrix** (plural **matrices**) is a rectangular array of numbers, symbols, or expressions (arranged in rows and columns) that is treated in certain prescribed ways.

In a simplified way we can say that it is a table of numbers.



Each element of a matrix is often denoted by a variable with two subscripts. For instance, $a_{2,1}$ represents the element at the second row and first column of a matrix A.

Elements or **entries** of a matrix are arranged in **rows** and **columns**,

e.g. a matrix A has elements a_{ij} (i-th row, j-th column), sometimes symbol A_{ij} is used

example: $$\mathbf{B} = \begin{bmatrix} 3 & -4 & 12 \\ 9 & 1 & 7 \\ 21 & 8 & 4 \end{bmatrix}$$

Matrix, matrices:

Matrices are commonly written in box brackets or an alternative notation uses large parentheses instead of box brackets:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

The specifics of symbolic matrix notation varies widely, with some prevailing trends (brackets).

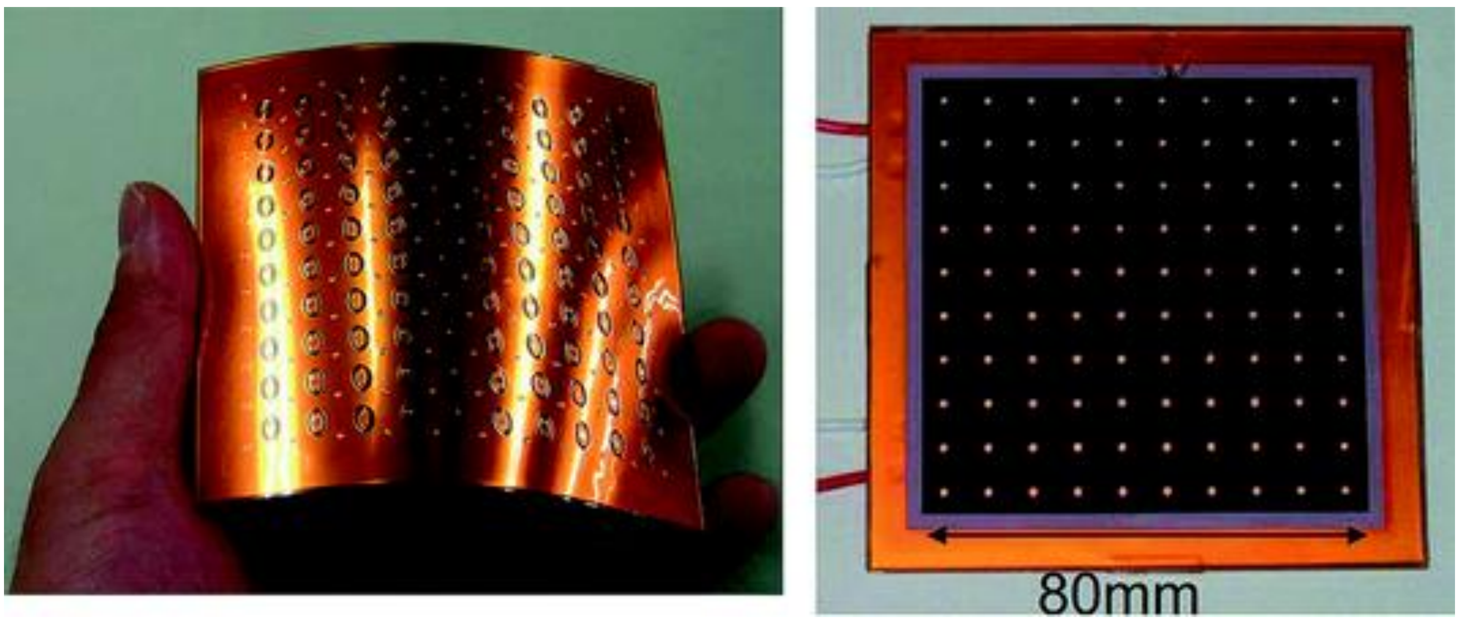
Matrices are usually symbolized using **upper-case letters** (sometimes in bold **A**, sometimes in normal style A), while the corresponding **lower-case letters**, with two subscript indices (e.g., a_{11}, or $a_{1,1}$), represent the elements (entries).

Matrices – why we use them? (application area):

In the digital world, matrices are everywhere...
(digital communication, visualisation, science, ...)

Matrices – why we use them? (application area):

A great variety of examples from science, where the approach involving matrices is involved (many types of scientific data-sets can be described and quantified by means of matrices).



- a sample of a new-technology solar panel material (flexible copper polyimide substrate).

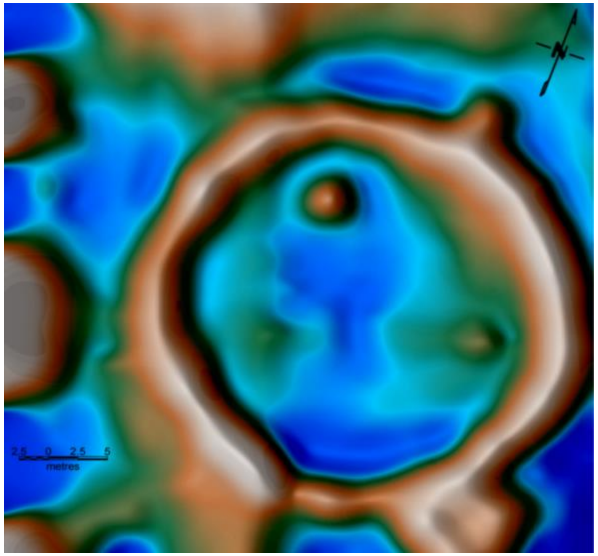
Another application of matrices is in the solution of systems of linear equations.

Matrices – why we use them? (application area):

Application in archaeological prospection – Earth magnetic field is measured, with the aim to detect old ditches (filled by humus with high concentration of colloidal magnetic minerals, a product of humus soil bacteries)

measurements

(so called magnetometer)



measured anomalous magnetic field

all the data are stored in a form of a grid (a matrix)

Matrix, matrices:

The **order** or **dimension** of a matrix is given as: number of rows x number of columns (e.g. 2x3).

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$

If these number are equal, we have a so called **square matrix** (e.g. 5x5).

$$\begin{bmatrix} 4 & 12 & -3 & 9 & 1 \\ -5 & 18 & 2 & 10 & 7 \\ 17 & 4 & 1 & 0 & -5 \\ 3 & 17 & 2 & 12 & 6 \\ 80 & 2 & 15 & -15 & 3 \end{bmatrix}$$

We can work also with matrices with only one row or only one column – in such a case we often speak about **vectors**

(row vector and column vector).

Name	Size	Example
Row vector	$1 \times n$	$\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$
Column vector	$n \times 1$	$\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$
Square matrix	$n \times n$	$\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$

Comment: There even exists a matrix with dimension 1x1 (can be used to save one number – e.g. some physical constant).

Matrix, matrices:

Main diagonal of a matrix is a collection of elements a_{ij}, where i = j. It runs from the top left-hand corner to the bottom right-hand corner.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The **antidiagonal** (sometimes **counterdiagonal**, **secondary diagonal**, or **minor diagonal**) is running in exactly opposite way - from the top right-hand corner to the bottom left-hand corner.

When the matrix is a square matrix with the dimension N, then it is a collection of elements a_{ij}, such that $i + j = N + 1$.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Main types of matrices (1/4):

If all entries outside the main diagonal are zero, A is called a **diagonal matrix**.

If all entries of A below the main diagonal are zero, A is called an **upper triangular matrix**. Similarly if all entries of A above the main diagonal are zero, A is called a **lower triangular matrix**.

Name	Example with *n* = 3
Diagonal matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
Lower triangular matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
Upper triangular matrix	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

A **zero matrix** or **null matrix** is a matrix with all its entries being zero.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The **identity matrix** I_n of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0. It is a square matrix, a special kind of diagonal matrix.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Main types of matrices (2/4):

Transpose matrix A^T is formed by turning rows into columns and vice versa: $(A^T)_{i,j} = A_{j,i}$.

In a simple way we can say that we flip (reflect) the elements of the matrix along its main diagonal:

$$\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Transpose

A square matrix whose transpose is equal to itself is called a **symmetric matrix**: $A^T = A$.

$$A = \begin{bmatrix} 1 & 1 & 4 & -1 \\ 1 & 5 & 0 & -1 \\ 4 & 0 & 21 & -4 \\ -1 & -1 & -4 & 10 \end{bmatrix}$$

A matrix, fulfilling the condition $A^T = -A$ is called a **skew-symmetric matrix**.

Main types of matrices (3/4):

Inverse matrix (A^{-1}):

Matrix B, fulfilling the condition $A\,B = B\,A = I_n$ is called an **inverse matrix** and its symbol is A^{-1}

(a simplier way is: $A\,A^{-1} = I_n$).

Method for the evaluation of an inverse matrix will be explained later on (during the exercise).

Orthogonal matrix:

A square matrix, fulfilling the condition $A^T = A^{-1}$ is so called **orthogonal matrix**.

Main types of matrices (4/4):

Submatrix:

A submatrix of a matrix is obtained by deleting any collection of rows and/or columns.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$submatrices$

Used e.g. during determinants evaluation (later in this lecture).

Basic operations with matrices (1/5):

- addition, scalar multiplication

Operation	Definition
Addition	The *sum* **A**+**B** of two *m*-by-*n* matrices **A** and **B** is calculated entrywise: $(\mathbf{A}+\mathbf{B})_{i,j} = \mathbf{A}_{i,j} + \mathbf{B}_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$.
Scalar multiplication	The product *c***A** of a number *c* ~~(also called a scalar in the parlance of abstract algebra~~) and a matrix **A** is computed by multiplying every entry of **A** by *c*: $(c\mathbf{A})_{i,j} = c \cdot \mathbf{A}_{i,j}$.

Basic operations with matrices (2/5) :

- addition, scalar multiplication

Operation	Example
Addition	$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$
Scalar multiplication	$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 8 & 2\cdot -3 \\ 2\cdot 4 & 2\cdot -2 & 2\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$

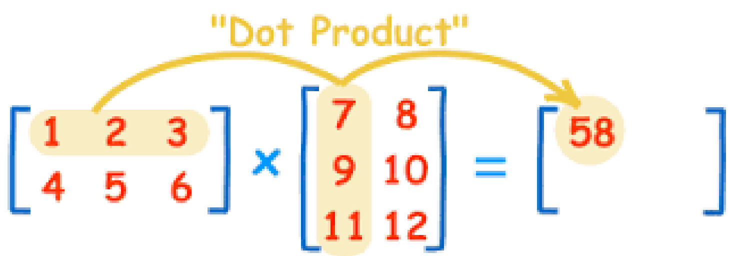
Basic operations with matrices (3/5) :

matrix multiplication

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix.

If A is an m-by-n matrix and B is an n-by-p matrix, then their matrix product AB is the m-by-p matrix, whose elements are given by **dot product** of the corresponding row of A and the corresponding column of B.

$$[\mathbf{AB}]_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,n}B_{n,j} = \sum_{r=1}^{n} A_{i,r}B_{r,j}.$$



example: $$\begin{bmatrix} \underline{2} & \underline{3} & \underline{4} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \underline{1000} \\ 1 & \underline{100} \\ 0 & \underline{10} \end{bmatrix} = \begin{bmatrix} 3 & \underline{2340} \\ 0 & 1000 \end{bmatrix}$$

For example, the underlined entry 2340 in the product is calculated as $(2 \times 1000) + (3 \times 100) + (4 \times 10) = 2340$.

Basic operations with matrices (4/5) :

matrix multiplication

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} =$$

Matrix multiplication **is not commutative** (!):

$$AB \neq BA \,.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$$

example: whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

Basic operations with matrices (5/5) :

matrix multiplication

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot1+2\cdot5 & 1\cdot2+2\cdot6 & 1\cdot3+2\cdot7 & 1\cdot4+2\cdot8 \\ 3\cdot1+4\cdot5 & 3\cdot2+4\cdot6 & 3\cdot3+4\cdot7 & 3\cdot4+4\cdot8 \\ 5\cdot1+6\cdot5 & 5\cdot2+6\cdot6 & 5\cdot3+6\cdot7 & 5\cdot4+6\cdot8 \end{pmatrix} = \begin{pmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \end{pmatrix}$$

Matrix multiplication **is not commutative** (!):

$$AB \neq BA \, .$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$$

example: whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

Lecture 1: Basic terms and rules in mathematics

Content:

- introduction (why so much Math in BCH?),
- repetition of logical operators,
- numbers, number systems,
- sets, basic operations with sets,
- matrices, operations with matrices,
- determinants.

Main operations with square matrices (1/3) :

Trace

The trace, tr(A) of a square matrix A is the sum of its diagonal entries.

Determinant

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

It gives us important information about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

The determinant of a matrix A is denoted det(A), det A, or |A|.

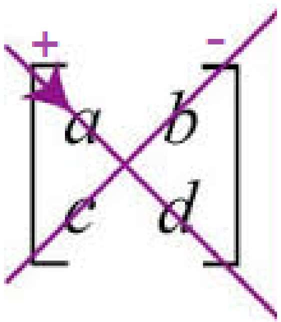
$$|\mathbf{B}| = \begin{vmatrix} 3 & -4 & 12 \\ 9 & 1 & 7 \\ 21 & 8 & 4 \end{vmatrix}$$

Main operations with square matrices (2/3) :

Determinant – evaluation:

Wellknown is the so called Rule of Sarus (or Sarrus' scheme).

For a 2 x 2 matrix it is valid:



$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

For a 3 x 3 matrix it is valid:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} =$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

Another effective way how to compute it (next slide):

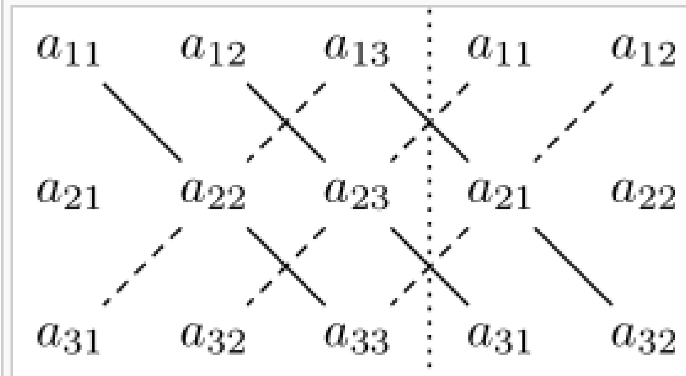
Main operations with square matrices (3/3) :

Determinant – evaluation:

The rule of Sarus (or Sarrus' scheme).

For a 3 x 3 matrix it is valid:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



Sarrus' rule: The determinant of the three columns on the left is the sum of the products along the solid diagonals minus the sum of the products along the dashed diagonals

$$\det(M) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$