Lecture 4: limits of functions, sequences, series

Content:

- limit of a function
- methods of limits evaluation
- continuous function
- sequences of real numbers
- series

Limit of a function - introduction

Limit of a function in some point speaks about special properties of a function and is very important in mathematical analysis.

Description (not a real definition):

If *f(x)* is some function then a limit of function *f* in point *a* is *L*:

$$\lim_{x \to a} f(x) = L$$

is to be read "the limit of *f(x)* as *x* approaches *a* is *L*".

(or in a very simple way "the limit of *f(x)* in *a* is *L* ")

It means that if we choose values of *x* which are close but not equal to *a*, then *f(x)* will be close to the value *L*;

moreover, *f(x)* gets closer and closer to *L* as *x* gets closer and closer to *a* (we can also say that f(x) converges to *L* for $x \to a$).

Comment: Point *a* can be also Infinity ($\pm\infty$).

Limit of a function - introduction

Example: If f(x) = x + 3 then

$$\lim_{x \to 4} x + 3 = 7$$

But this is a <u>very simple example</u> and for such situations we really do not need the whole concept of limits evaluation in mathematics.

We should inspect more special situations.

Example: If f(x) = sin(x)/x

$$\lim_{x \to 0} \frac{\sin(x)}{x} = ?$$

x	$\frac{\sin x}{x}$
1	0.841471...
0.1	0.998334...
0.01	0.999983...

This is not so a simple example, because when we substitute x=0 then we get a expression of 0/0 type, which does not exists.

But there is a solution and we will come to it (later on).

Limit of a function - introduction

Next example:

Unfortunately, substituting numbers can sometimes suggest a wrong answer.

..."*x* close to *a*„ – but how close is close enough?

Suppose we had taken the function:

$$\lim_{x \to 0} \frac{101000x}{100000x + 1} = ?$$

Substitution of some "small values of x" could lead us to believe that the limit is 1.
Only when we substitute very small values, we realize that the limit is 0 (zero)!

Limit of a function - introduction

$$\lim_{x \to 0} \frac{101000x}{100000x + 1} = 0$$

x	101000x/(100000x + 1)
1	1.0100
0.1	1.0099
0.01	1.0090
0.001	1.0000
0.0001	0.9182
0.00001	0.5050
10^{-6}	0.0918
10^{-7}	0.0100
10^{-8}	0.0010
10^{-9}	0.0001

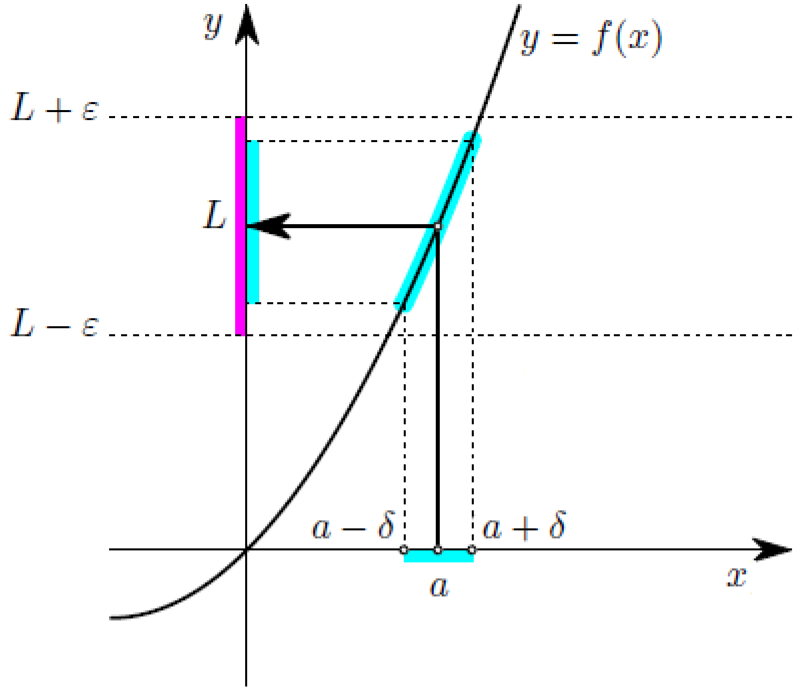
Limit of a function:

Definition: We say that L is the limit of *f(x)* as $x \to a$, if:

(1) *f(x)* need not be defined at *x* = *a*, but it must be defined for all other *x* in some interval which contains *a*.

(2) for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all *x* in the domain of *f(x)* one has:

$$|x - a| < \delta \ implies \ |f(x) - L| < \varepsilon.$$



Limit of a function:

Definition: We say that L is the limit of *f(x)* as $x \to a$, if:

(1) *f(x)* need not be defined at $x = a$, but it must be defined for all other *x* in some interval which contains *a*.

(2) for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all *x* in the domain of *f(x)* one has:

$$|x - a| < \delta \ implies \ |f(x) - L| < \varepsilon.$$

Why the absolute values? The quantity $|x - a|$ is the distance between the points *x* and *a* on the number line, and one can measure how close *x* is to *a* by calculating $|x - a|$. The inequality $|x - a| < \delta$ says that "the distance between *x* and *a* is less than δ," or that "*x* and *a* are closer than δ. "

Parameters δ and ε are also called as surroundings of points *a* and *L*, respectively.

Limit of a function: $|x - a| < \delta \; implies \; |f(x) - L| < \varepsilon.$

Evaluation of a limit, based on its definition.

Example: $$\lim_{x \to 5}(2x + 1) = 11$$

Solution:

We have *f(x)* = 2*x* + 1, *a* = 5 and *L* = 11, and the question we must answer is: "how close should *x* be to 5 if want to be sure that *f(x)* = 2*x* + 1 differs less than ε from *L* = 11?"

To figure this out we try to get an idea of how big |*f(x)* – *L*| is:

$|f(x) - L| = |(2x + 1) - 11| = |2x - 10| = 2|x - 5| = 2|x - a|$.

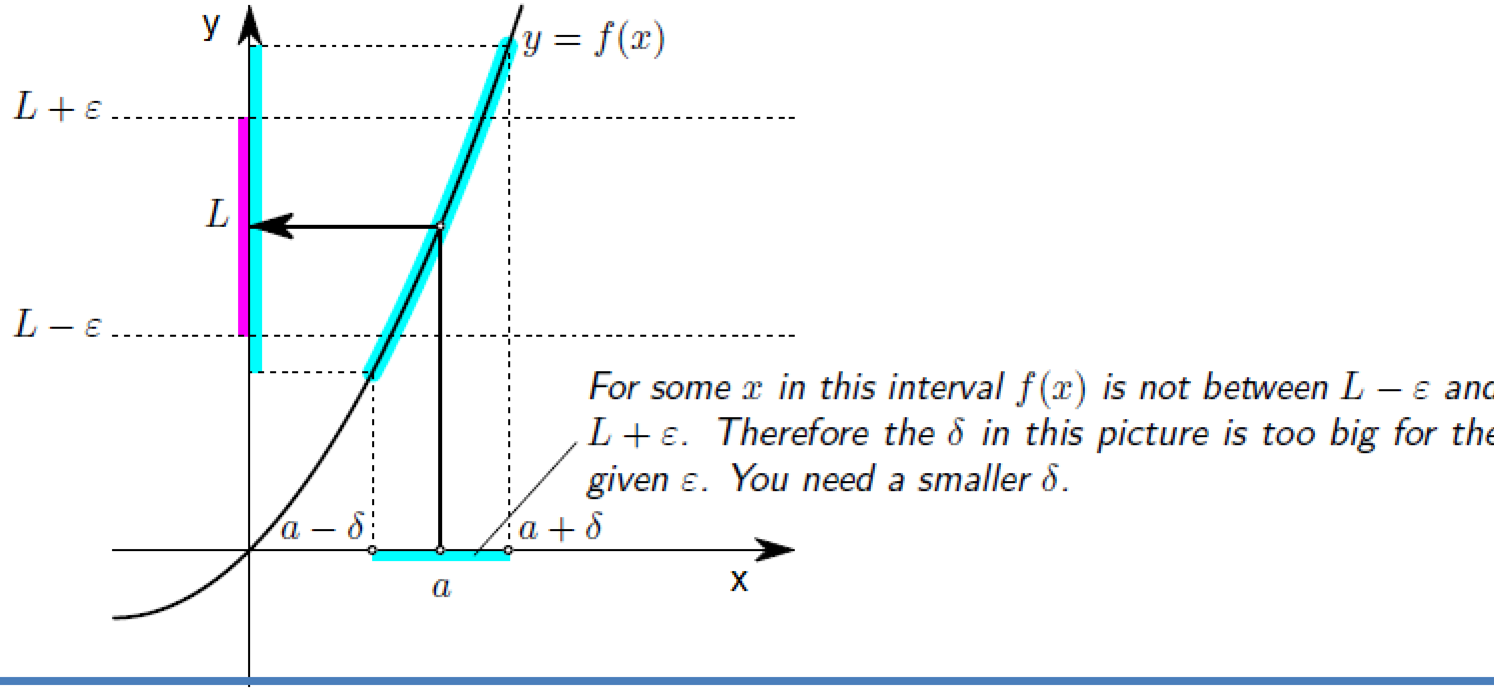
So, if $2|x - a| < \varepsilon$ then we have $|f(x) - L| < \varepsilon$, i.e.

if $|x - a| < 1/2\varepsilon$ then $|f(x) - L| < \varepsilon$.

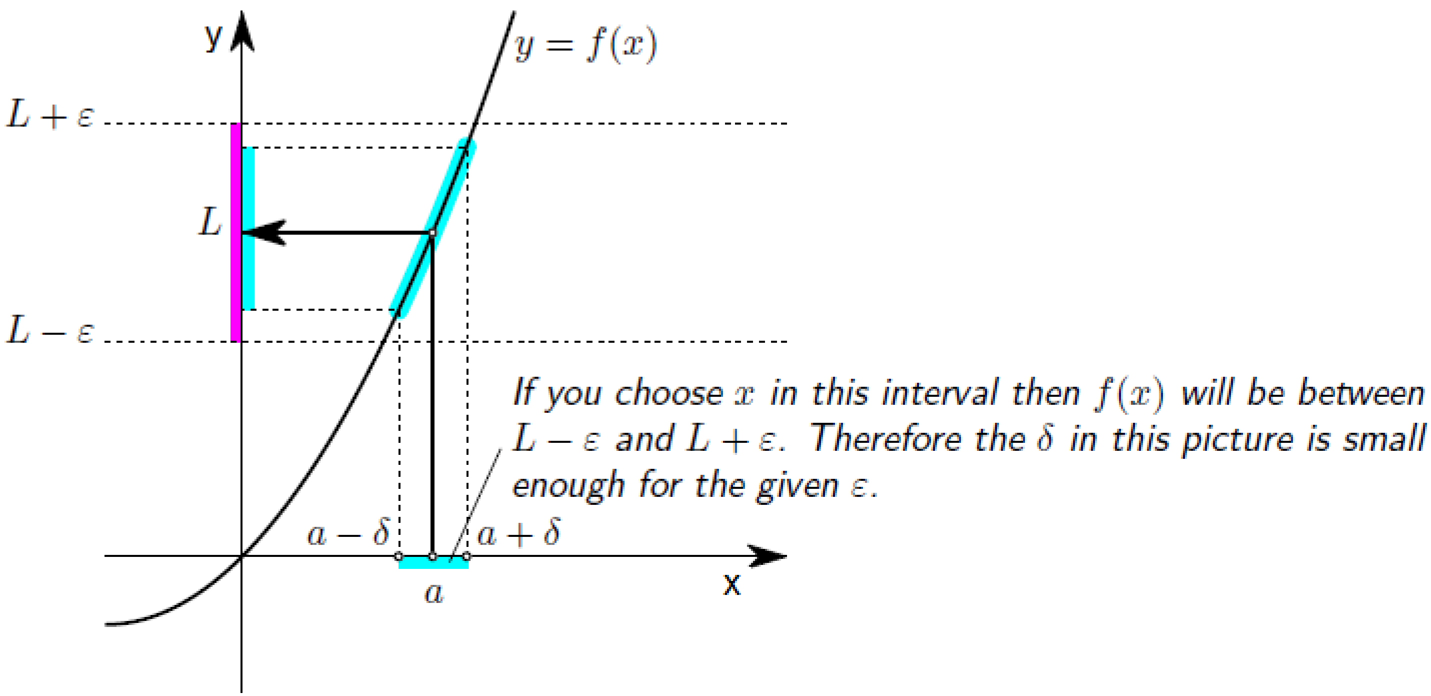
We can therefore choose $\delta = 1/2\varepsilon$. No matter what $\varepsilon > 0$ we are given our δ will also be positive, and if $|x - 5| < \delta$ then we can guarantee $|(2x + 1) - 11| < \varepsilon$.

That shows that $\lim_{x \to 5} (2x + 1) = 11$.

This kind of solution is quite cumbersome, so we have to introduce some more efficient ways how to evaluate limits.



y
$y = f(x)$
$L + \varepsilon$
L
$L - \varepsilon$
For some x in this interval $f(x)$ is not between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is too big for the given ε. You need a smaller δ.
$a - \delta$
$a + \delta$
a
x



y
$y = f(x)$
$L + \varepsilon$
L
$L - \varepsilon$
If you choose x in this interval then $f(x)$ will be between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is small enough for the given ε.
$a - \delta$
$a + \delta$
a
x

Methods of limits evaluation:

1. Substitution method
2. Factoring method
3. Conjugate method
4. Division method
5. L'Hospital's Rule

Comment: Rational function $f(x) = P_n(x)/Q_n(x)$, where $P_n(x)$ and $Q_n(x)$ are polynomials [$Q_n(x)$ is a nonzero polynomial].

$$\frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_0}$$

Methods of limits evaluation:

1. Substitution method:

Just simply put the value for *x* into the expression.

Simple examples:

$$\lim_{x \to 10} \frac{x}{2} = \frac{10}{2} = 5$$

$$\lim_{x \to -1} \frac{x^2 + 4x + 3}{x} = \frac{1 - 4 + 3}{-1} = \frac{0}{-1} = 0$$

But what to do in following cases?:

$$\lim_{x \to 3} \frac{x + 4x}{x - 3} = \frac{3 + 12}{0} = \frac{15}{0}$$

$$\lim_{x \to \infty} \frac{15}{x + 3} = \frac{15}{\infty}$$

Specific case:

What will happen when we have to solve a limit, where we get finally an expressions of type 1/∞?

In fact 1/∞ is known to be undefined, because strictly speaking Infinity is not a number, it is an idea. But we can approach it.

x	$\frac{1}{x}$
1	1,00000
2	0,50000
4	0,25000
10	0,10000
100	0,01000
1.000	0,00100
10.000	0,00010

So - the limit of 1/x as x approaches Infinity is 0.

And what will happen when we take the exactly opposite case – expression of type 1/0?

Exactly the opposite situation (beside the fact that also this is undefined expression): The limit of 1/x as x approaches 0 is Infinity.

Next specific case:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

This is a so called indeterminate expression (form)

(these are expressions of type 0/0 or ∞/∞).

We will come to a general solution method for this kind of limits later on.

Methods of limits evaluation:

2. Factoring method:

Factoring – decomposition to factors, e.g.: $(x^2-1)=(x+1)(x-1)$

Example from previous slide:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}=\lim_{x\to 1}(x+1)=1+1=2$$

This method is mainly suitable for so called rational functions limits evaluation.

Next example:

$$\lim_{x\to -1}\frac{x^2+4x+3}{x+1}=?$$

Methods of limits evaluation:

3. Conjugate method:

Also for rational functions – sometime it helps, when we multiply the nominator and denominator of the fraction with a conjugate. Conjugate – in the case of binomials it is formed by negating the second term of the binomial (e.g. the conjugate of x+y is x-y).

Example:

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4-x}=\lim_{x\to 4}\frac{\left(2-\sqrt{x}\right)}{\left(4-x\right)}\frac{\left(2+\sqrt{x}\right)}{\left(2+\sqrt{x}\right)}=\lim_{x\to 4}\frac{\left(2^2+2\sqrt{x}-2\sqrt{x}-x\right)}{\left(4-x\right)\left(2+\sqrt{x}\right)}=$$

$$=\lim_{x\to 4}\frac{\left(4-x\right)}{\left(4-x\right)\left(2+\sqrt{x}\right)}=\lim_{x\to 4}\frac{1}{2+\sqrt{x}}=\frac{1}{2+\sqrt{4}}=\frac{1}{4}$$

Methods of limits evaluation:

4. Division method:

Valid only for limits of rational functions with $x \to \infty$.

$$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \lim_{x\to\infty}\frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_0}$$

1. option $\to$ $\mathbf{n > m}$ $\quad \lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \infty$

2. option $\to$ $\mathbf{n < m}$ $\quad \lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = 0$

3. option $\to$ $\mathbf{n = m}$ $\quad \lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \frac{a_n}{b_m} \to$ **Division of coefficient of the largest powers**

Methods of limits evaluation:

4. Division method:

Valid only for limits of rational functions with $x \to \infty$.

Solution is based on the division of all terms of both polynomials (in nominator and denominator) with the highest power of x.

Examples:

$$\lim_{x\to\infty}\frac{3x+1}{2x+5}=\lim_{x\to\infty}\frac{3x/x+1/x}{2x/x+5/x}=\lim_{x\to\infty}\frac{3+1/x}{2+5/x}=\frac{3+0}{2+0}=\frac{3}{2}$$

$$\lim_{x\to\infty}\frac{x^3+3x}{5x^3+2x^2+8}=\lim_{x\to\infty}\frac{x^3/x^3+3x/x^3}{5x^3/x^3+2x^2/x^3+8/x^3}=\lim_{x\to\infty}\frac{1+3/x^2}{5+2/x+8/x^3}=\frac{1}{5}$$

$$\lim_{x\to\infty}\frac{x^2}{x^3+1}=\lim_{x\to\infty}\frac{x^2/x^3}{x^3/x^3+1/x^3}=\lim_{x\to\infty}\frac{1/x}{1+1/x^3}=\frac{0}{1+0}=0$$

Methods of limits evaluation:

5. L'Hospital's Rule:

Valid for limits of so called indeterminate expressions (forms) (expressions of type 0/0 or ∞/∞).

This rule is using derivatives, so we will return to it later during the term (future lectures).

Evaluation of limits for expressions:

All basic operations (+, - ,* , /) have a simple position in the evaluation of limits:

(limit of an addition of two expressions is equal to the addition of these two limits,... etc.)

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=A+B$$

$$\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=A-B$$

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=A\cdot B$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}=\frac{A}{B}\qquad B\neq 0$$

Left and right limits:

When we let "*x* approach *a*" we allow *x* to be both larger or smaller than *a*, as long as *x* gets close to *a*.

If we explicitly want to study the behaviour of *f(x)* as *x* approaches *a* through values larger (lower) than *a*, then we write

a right-limit (or limit from the right-hand side):

$$\lim_{x \searrow a} f(x) \text{ or } \boxed{\lim_{x \to a+} f(x)} \text{ or } \lim_{x \to a+0} f(x) \text{ or } \lim_{x \to a, x > a} f(x)$$

and a left-limit (or limit from the left-hand side):

$$\lim_{x \nearrow a} f(x) \text{ or } \boxed{\lim_{x \to a-} f(x)} \text{ or } \lim_{x \to a-0} f(x) \text{ or } \lim_{x \to a, x < a} f(x)$$

All four notations are in use (in various text-books).

Lecture 4: limits of functions, sequences, series

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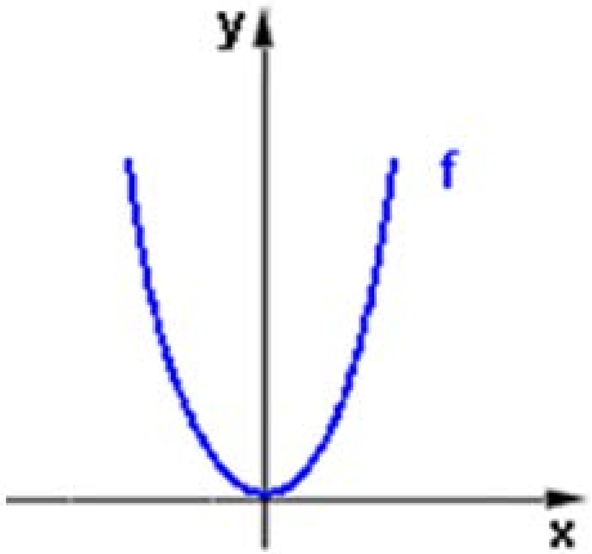
Relation of the limit of a function to continuity:

The notion of the limit of a function is very closely related to the concept of continuity.

Definition: A function *f(x)* is said to be continuous at *a* if it is both defined at *a* and its value at *a* equals the limit of f(x) as *x* approaches *a*:

$$\lim_{x \to a} f(x) = f(a)$$

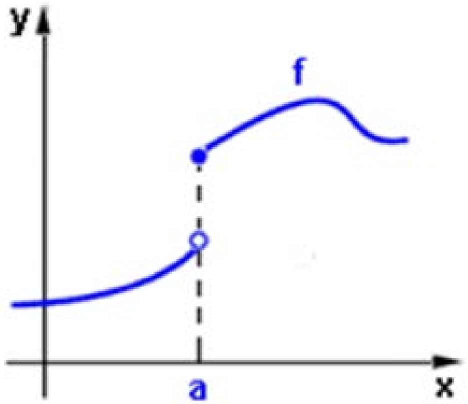
In other words: a continuous function is smooth, without any “steps”.



example of a continuous function.

Discontinuous function *f(x)* is a function, which for certain values or between certain values of the variable *x* does not vary continuously as the variable *x* increases or decreases.

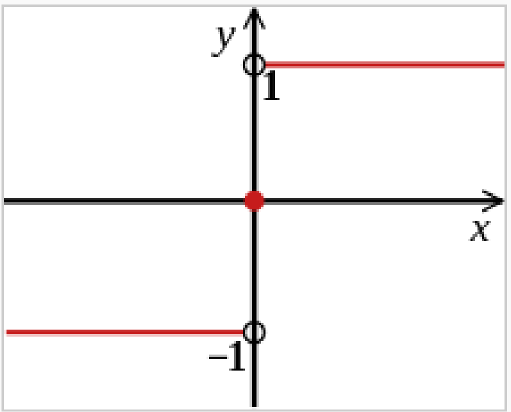
In other words: a discontinuous function can have "steps".



example of a discontinuous function.

Example: the so called signum or sign function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Plot of the signum function. The hollow dots indicate that sgn(*x*) is 1 for all *x*>0 and −1 for all *x*<0.

Relation of the limit of a function to continuity:

For continuous functions it must be valid that the left-limit is equal to the right-limit (this is valid for the majority of cases):

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

For discontinuous functions this condition is invalid, the left-limit is not equal to the right-limit:

$$\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$$

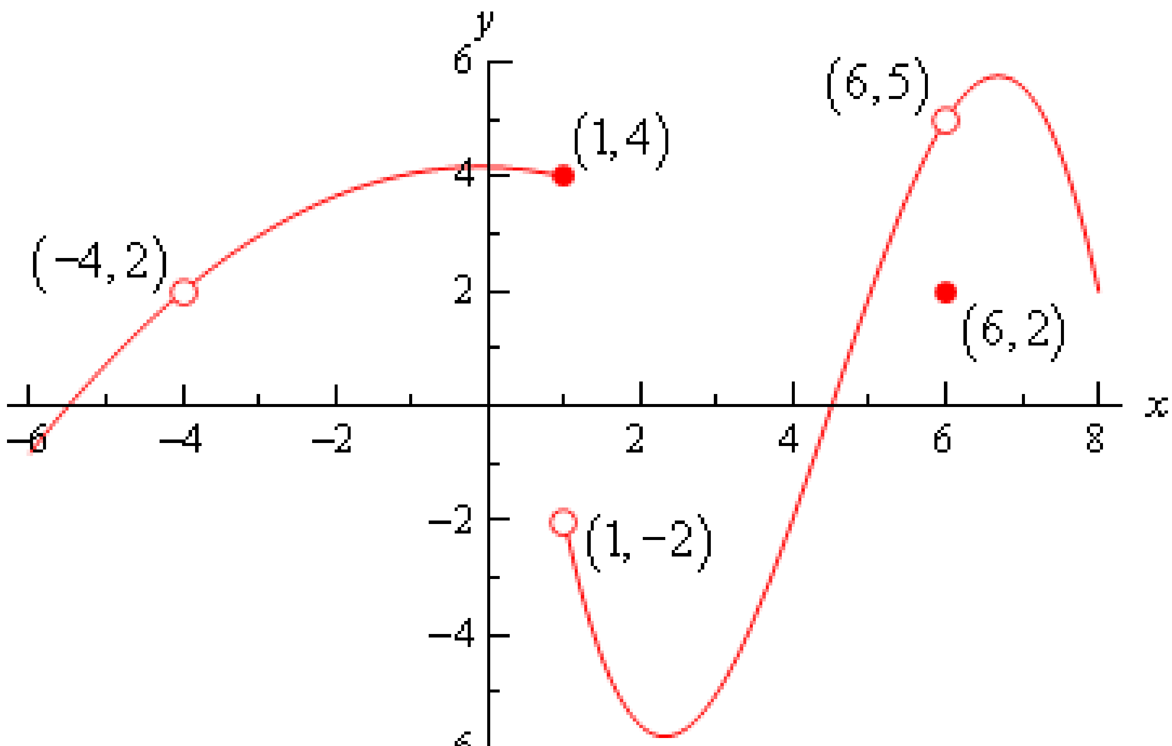
Example (discontinuous function):

$$\lim_{x \to 0+} \frac{|\sin x|}{\sin x} = 1, \quad \text{but} \quad \lim_{x \to 0-} \frac{|\sin x|}{\sin x} = -1.$$

Relation of the limit of a function to continuity:

For continuous functions it must be valid that the left-limit is equal to the right-limit (this is valid for the majority of cases):

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$



... but not for this one (x=6)

Lecture 4: limits of functions, sequences, series

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Sequences of (real) numbers

Description: Sequence is an ordered collection of elements (numbers) from a set in which repetitions are allowed.

Like a set, it contains members (also called elements, or terms).

The number of elements (possibly infinite) is called the length of the sequence.

The usual notation for a real number sequence is:

$$\{a_n\}_{n=1}^{M} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty} \quad \text{or} \quad a_1, a_2, \ldots$$

(n is a natural number, giving the sequence number of an element)

The total number of elements (length) can be a finite number (e.g. M) or Infinity (∞) – from this point of view we recognize finite and infinite sequence.

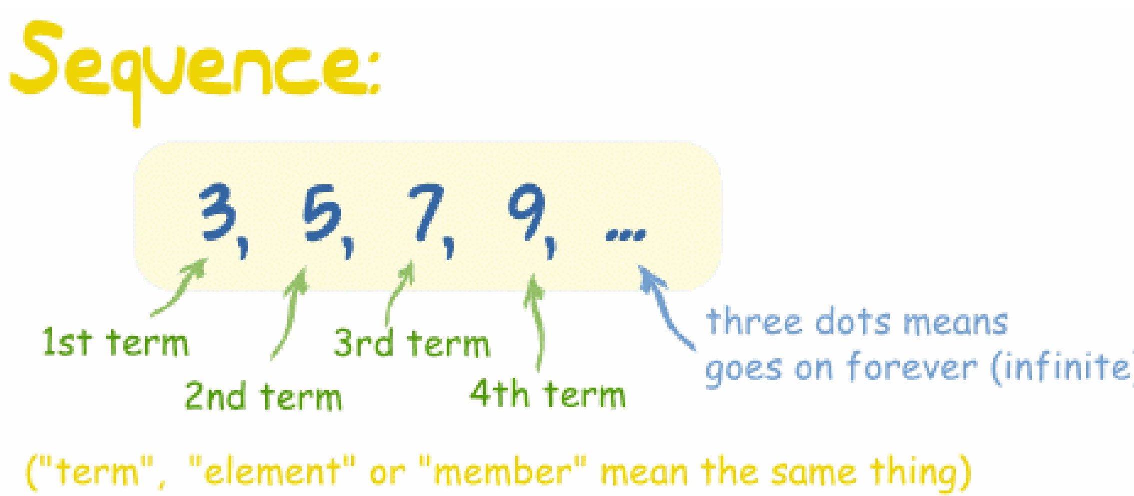
Sequences of (real) numbers

Formal definition:

A sequence is a function from a subset of the natural numbers to the real numbers set. In other words, a sequence is a map f(n) : N → R.
We can simply write - for all n is valid a_n : N → R.

Indexing: The terms of a sequence are commonly denoted by a single variable, say a_n, where the index *n* indicates the *n*th element of the sequence.

a_1	$\leftrightarrow$	1st element
a_2	$\leftrightarrow$	2nd element
a_3	$\leftrightarrow$	3rd element
$\vdots$		$\vdots$
a_{n-1}	$\leftrightarrow$	(n-1)th element
a_n	$\leftrightarrow$	nth element
a_{n+1}	$\leftrightarrow$	(n+1)th element
$\vdots$		$\vdots$



Sequences of (real) numbers

1. A sequence can be given by the list of its terms:

$a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 7, a_5 = 11, \ldots$

2. A sequence may be also defined by giving an explicit formula for the n^{th} term.

E.g.:

$$a_n = \frac{1}{n}, \quad n = 1, 2, \ldots$$

it is a sequence of terms: $a_1 = 1, a_2 = 1/2, a_3 = 1/3, \ldots$

3. A sequence may also be defined inductively - by recursion.

E.g.:

$$a_1 = 0, a_2 = 1, a_{n+2} = \frac{a_n + a_{n+1}}{2}, \quad n = 1, 2, \ldots$$

it is a sequence of terms: $a_1 = 0, a_2 = 1, a_3 = 1/2, a_4 = 3/4, \ldots$

Sequences of (real) numbers

How to find "the next number" in a sequence?

There exist several rules that work, but no one of them is a general rule and very often a „trial-and-error method“ must be used.

One interesting method - based on finding differences (or divisions) between each pair of terms.

Example:

What is the next number in the sequence 7, 9, 11, 13, 15, ... ?

Solution: The differences are always 2,

so we can guess that "2n" is part

of the answer. Let us try 2n.

Such a model is wrong by 5,

So the right answer is:

$$a_n = 2n + 5.$$

n:	1	2	3	4	5
Terms :	7	9	11	13	15
2n:	2	4	6	8	10
Wrong by:	5	5	5	5	5

Some very important kind of sequences:

1. Arithmetic sequence:

In an Arithmetic Sequence the difference between one term and the next is a constant. In other words, we just add the same value each time ... infinitely.

Example: 1, 4, 7, 10, 13, 16, 19, 22, 25, ...

This sequence has a difference of 3 between each number.

In general we can write an arithmetic sequence in a form:

$$\{a, a+d, a+2d, a+3d, \dots\}$$

where:

a is the first term, and

d is the difference between the terms (called the "common difference")

We can write an arithmetic sequence as a rule:

$$a_n = a + d(n-1)$$

(we use here "n–1" because d is not used in the 1st term).

Some very important kind of sequences:

2. Geometric sequence:

In a Geometric Sequence each term is found by multiplying the previous term by a constant.

Example: 2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number. Each term (except the first term) is found by multiplying the previous term by 2.

In general we write a geometric sequence in a form:

$$\{a, aq, aq^2, aq^3, \dots\}$$

where:

a is the first term, and

q is the factor between the terms (called common ratio or quotient)

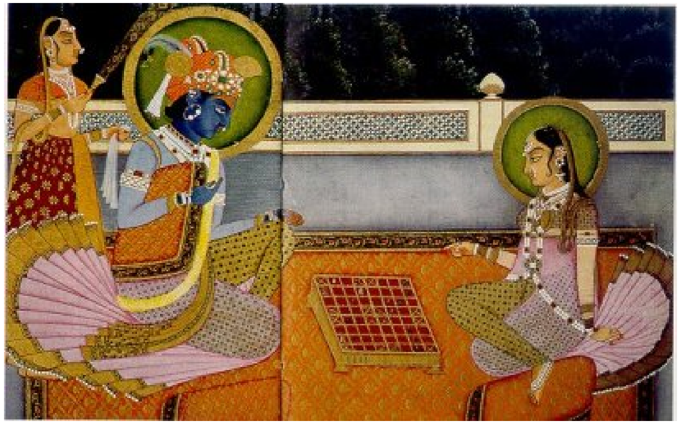
(q can not be equal zero - we would get a sequence {a,0,0,...})

Also here, we can write an geometric sequence as a rule:

$$a_n = aq^{n-1}$$

(we use "n-1" because aq^0 is for the 1^{st} term).

2. Geometric sequence:

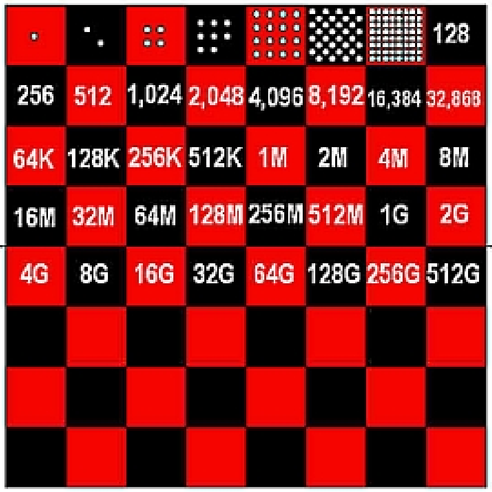


Legend of Paal Paysam (1/2)

The legend goes that the tradition of serving Paal Paysam to visiting pilgrims started after a game of chess between the local king and the lord Krishna himself:

The king was a big chess enthusiast and had the habit of challenging wise visitors to a game of chess. One day a traveling sage was challenged by the king. To motivate his opponent the king offered any reward that the sage could name. The sage modestly asked just for a few grains of rice in the following manner: the king was to put a single grain of rice on the first chess square and double it on every consequent one.

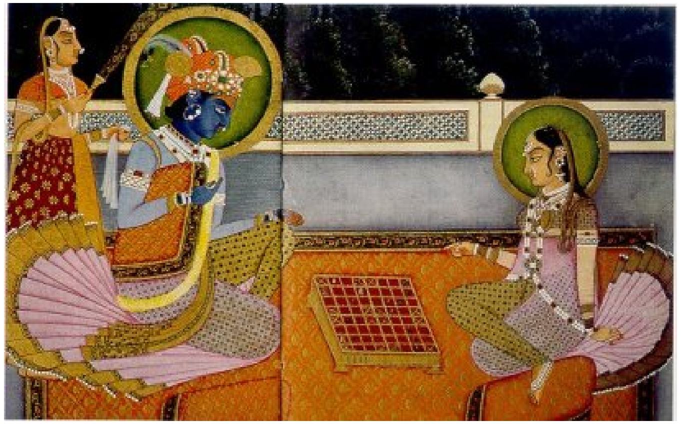
Having lost the game and being a man of his word the king ordered a bag of rice to be brought to the chess board. Then he started placing rice grains according to the arrangement:
1 grain on the first square, 2 on the second, 4 on the third, 8 on the fourth and so on:



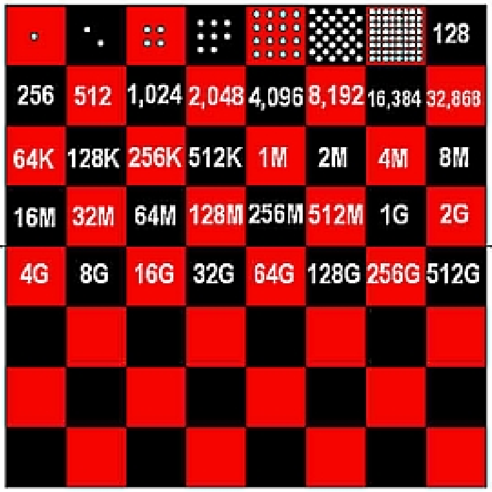
2. Geometric sequence:

Legend of Paal Paysam (2/2)

The legend goes that the tradition of serving Paal Paysam to visiting pilgrims started after a game of chess between the local king and the lord Krishna himself:



Following the exponential growth of the rice payment the king quickly realized that he was unable to fulfill his promise because on the twentieth square the king would have had to put 1,000,000 grains of rice. On the fortieth square the king would have had to put 1,000,000,000 grains of rice. And, finally on the sixty fourth square the king would have had to put more than 18,000,000,000,000,000,000 grains of rice which is equal to about 210 billion tons and is allegedly sufficient to cover the whole territory of India with a meter thick layer of rice.

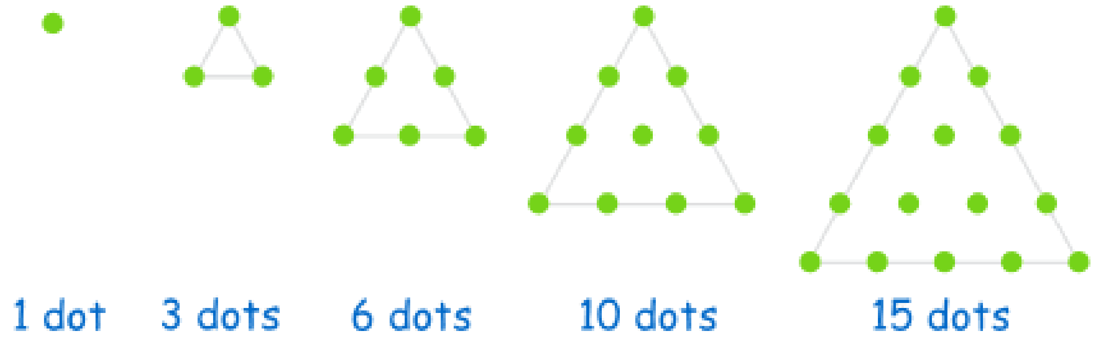


Some interesting kind of sequences:

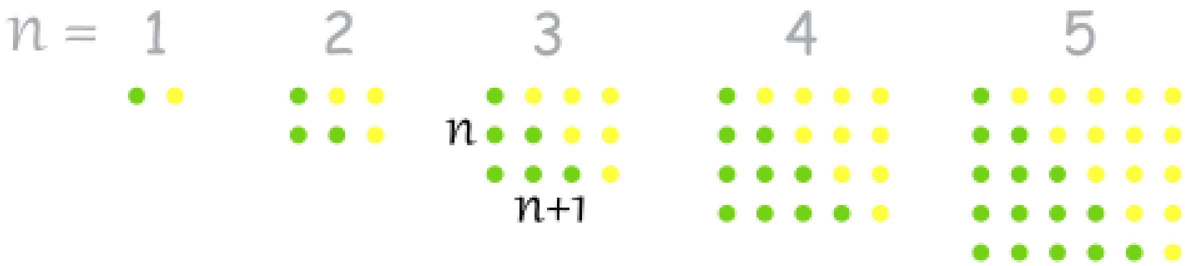
3. So called triangular number sequence :

1, 3, 6, 10, 15, 21, 28, 36, 45, ...

This sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence:



Finding the rule - rearranging the dots and form them into a rectangle:

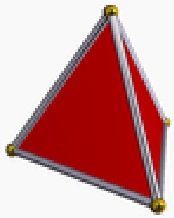
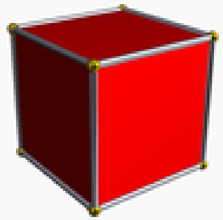
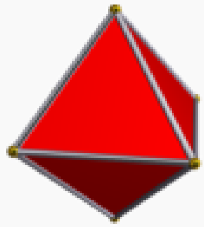
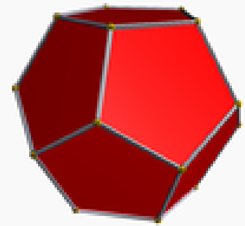
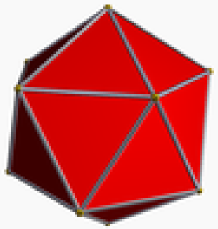


we get finally: $a_n = n(n+1)/2$

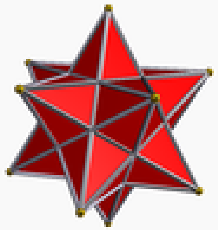
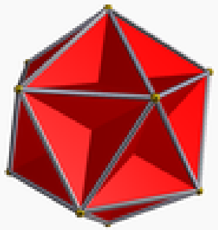
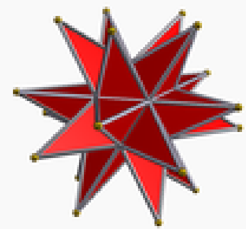
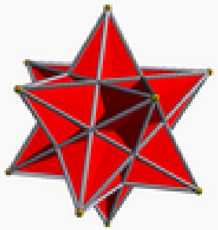
triangular number sequence :

and where we can use this?

Platonic solids

				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}
χ = 2	χ = 2	χ = 2	χ = 2	χ = 2

Kepler-Poinsot polyhedra

			
Small stellated dodecahedron {5/2, 5}	Great dodecahedron {5, 5/2}	Great stellated dodecahedron {5/2, 3}	Great icosahedron {3, 5/2}
χ = −6	χ = −6	χ = 2	χ = 2

shapes of basic regular polyhedral bodies

Properties of sequences:

Increasing and decreasing:

A sequence is said to be monotonically increasing if each consecutive term is greater than or equal to the one before it.

If each consecutive term is strictly greater than (>) the previous term then the sequence is called strictly monotonically increasing.

A sequence is monotonically decreasing if each consecutive term is less than or equal to the previous one.

If each consecutive term is strictly smaller than (<) the previous term then the sequence is called strictly monotonically decreasing.

If a sequence is either increasing or decreasing it is called a monotone sequence.

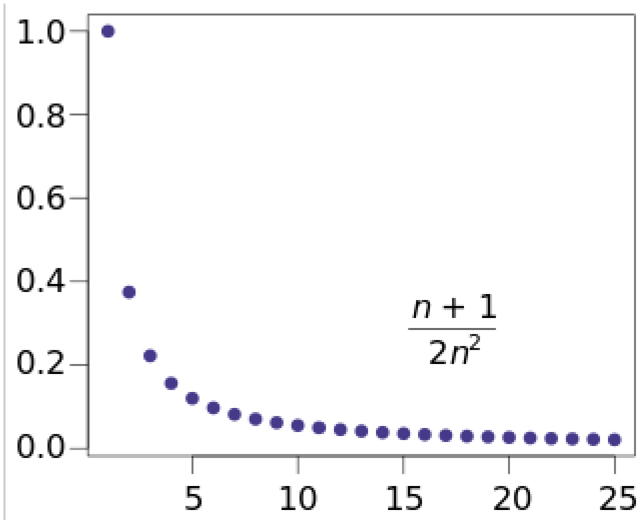
Properties of sequences:

Limits and convergence:

One of the most important properties of a sequence is convergence. Informally, a sequence converges if it has a limit – a sequence has a limit if it approaches some value *L*, called the limit, as *n* becomes very large:

$$\lim_{n \to \infty} a_n = L$$

example:



More precisely - the sequence converges if there exists a limit *L* such that the remaining a_n terms are arbitrarily close to *L* for some *n* large enough.

Properties of sequences:

Limits and convergence:

If a sequence converges to some limit, then it is convergent; otherwise it is divergent.

If a_n gets arbitrarily large as $n \to \infty$ we write

$$\lim_{n\to\infty} a_n = \infty.$$

In this case we say that the sequence (a_n) *diverges*, or that it converges to infinity.

If a_n becomes arbitrarily "small" negative numbers (large in magnitude) as $n \to \infty$ we write

$$\lim_{n\to\infty} a_n = -\infty$$

and say that the sequence diverges or converges to minus infinity.

Lecture 4: limits of functions, sequences, series

Content:

- limit of a function
- methods of limits evaluation
- continuous function
- sequences of real numbers
- series

Series

Description: Informally speaking, series is the sum of the terms of a sequence:

$$
\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_N &= a_1 + a_2 + a_3 + \cdots \\
&\vdots
\end{aligned}
$$

We can also write the nth term of the series as:

$$S_N = \sum_{n=1}^{N} a_n$$

Concepts used to talk about sequences, such as convergence, limits,.. carry over to series.

E.g. a limit (for a converging sequence):

$$\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n$$

Series

1. Arithmetic series:

Summing an arithmetic series:

To sum up the terms of arithmetic sequence:

a + (a+d) + (a+2d) + (a+3d) + ...

use this formula:

$$\sum_{k=0}^{n-1}(a+kd)=\frac{n}{2}\left[2a+(n-1)d\right]$$

where d is the common difference and n the length of the sequence.

1. Summing an arithmetic sequence - example:

Example: Add up the first 10 terms of the arithmetic sequence:

{ 1, 4, 7, 10, 13, ... }

The values of **a**, **d** and **n** are:

- **a** = **1** (the first term)
- **d** = **3** (the "common difference" between terms)
- **n** = **10** (how many terms to add up)

So:

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n}{2}(2a+(n-1)d)$$

Becomes:

$$\sum_{k=0}^{10-1}(1+k\cdot 3) = \frac{10}{2}(2\cdot 1+(10-1)\cdot 3)$$

$$= 5(2+9\cdot 3) = 5(29) = 145$$

2. Geometric series:

Summing a geometric series:

To sum up the terms of geometric sequence:

$$a + aq + aq^2 + ... + aq^{n-1}$$

use this formula (valid for $q \neq 1$):

$$\sum_{k=0}^{n-1} aq^k = a\left(\frac{1-q^n}{1-q}\right)$$

where a is the first term, q the common ratio or quotient

and n the length of the sequence.

Important: This formula can be simplified for converging inf. series, when the parameter q fulfils the following condition: $|q| < 1$.

$$\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q}$$

2. Summing a geometric sequence - example:

Example: Sum the first 4 terms of

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number.

The values of **a**, **r** and **n** are:

- **a = 10** (the first term)
- **r = 3** (the "common ratio")
- **n = 4** (we want to sum the first 4 terms)

So:

$$\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

Becomes:

$$\sum_{k=0}^{4-1}(10 \cdot 3^k) = 10\left(\frac{1-3^4}{1-3}\right) = 400$$

You can check it yourself:

10 + 30 + 90 + 270 = 400