Lecture 6: differentiation – further topics, complex numbers

Content:

- derivatives utilization - Taylor series
- derivatives utilization - L'Hospital's rule
- functions graph course analysis

- complex numbers, introduction
- complex numbers, basic operations
- complex numbers, functions

Derivatives utilization – Taylor series

Definition: The Taylor series of a real or complex-valued function *f* (x) that is infinitely differentiable at a real or complex number *a* is the power series:

$$\boxed{f(x) \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .}$$

In other words: a function *f* (x) can be approximated in the close vicinity of the point *a* by means of a power series, where the coefficients are evaluated by means of higher derivatives evaluation (divided by corresponding factorials).

Taylor series can be also written in the more compact sigma notation (summation sign) as:

$$\boxed{f(x) \doteq \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Comment: factorial 0! = 1.

Derivatives utilization – Taylor series

When *a* = 0, the Taylor series is called a Maclaurin series

(Taylor series are often given in this form):

$$f(x) \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

$\downarrow$ *a* = 0

$$\boxed{f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.}$$

... and in the more compact sigma notation (summation sign):

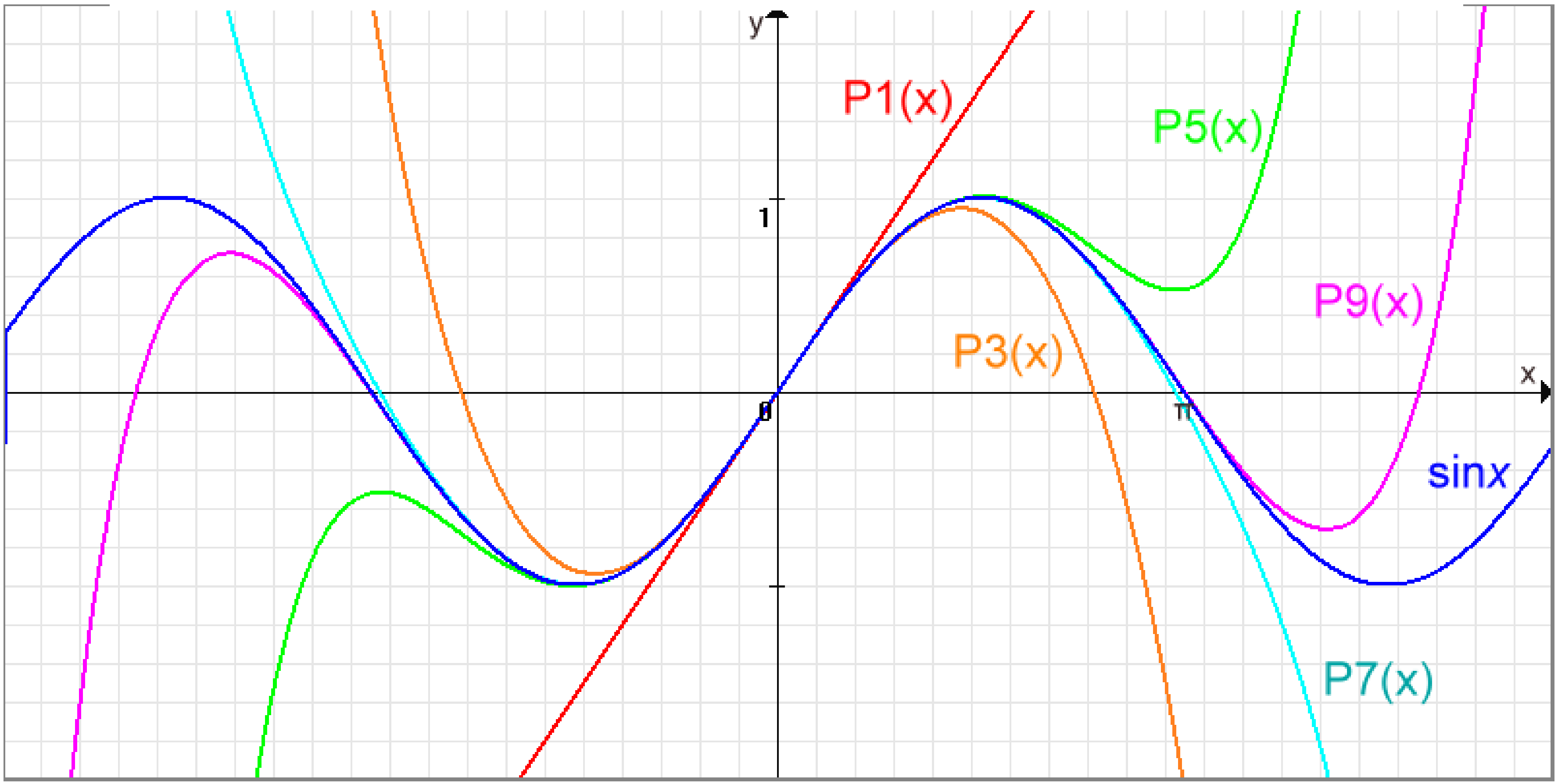
$$f(x) \doteq \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\downarrow$ *a* = 0

$$\boxed{f(0) \doteq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

graphical demonstration – approximation with polynomials (sin$x$)

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$



nice examples:

https://en.wikipedia.org/wiki/Taylor_series

http://mathdemos.org/mathdemos/TaylorPolynomials/

Derivatives utilization – Taylor (Maclaurin) series

Examples (Maclaurin series):

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Derivatives utilization – Taylor (Maclaurin) series

Taylor (Maclaurin) series are utilized in various areas:

1. Approximation of functions – mostly in so called local approx. (in the close vicinity of point *a* or 0).
   Precision of this approximation can be estimated by so called remainder or residual term R_n, describing the contribution of ignored higher order terms:

$$f(x) \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \boxed{R_n(x)}$$

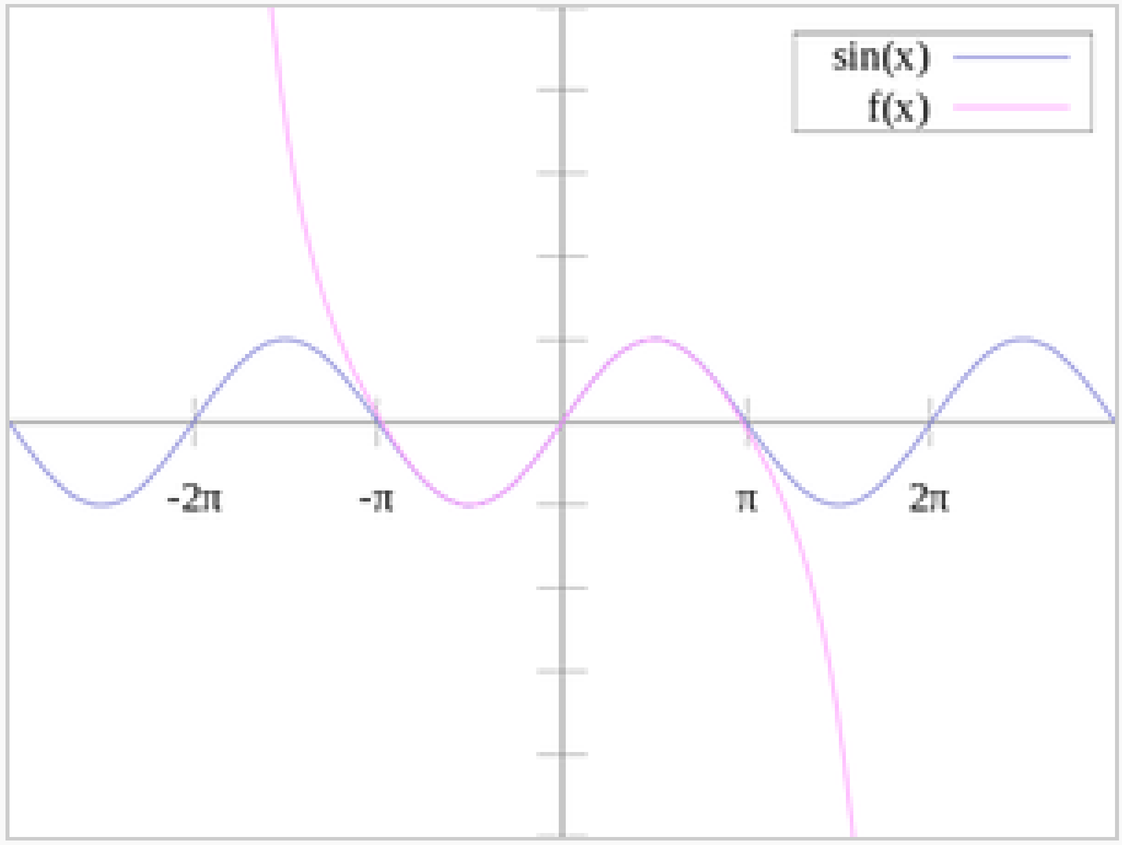
2. Simplification of solutions – so called linearization: in some solutions (|x| < 1) the higher degree terms can be ignored and only the constant and linear term are used (we do not have to work later on with more complicated higher degree polynomials).

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \cancel{\frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3} + \cdots .$$

Derivatives utilization – Taylor (Maclaurin) series

Local approximation of functions (in local vicinity of an isolated point, not on whole interval):

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$



The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full period centered at the origin.

The error in this approximation (for $-1 < x < 1$) is less than 0.000003.

Derivatives utilization – Taylor (Maclaurin) series

Comment: Derivation of Euler's formulas (complex numbers topic):

$e^{i\varphi} = \cos\varphi + i\sin\varphi$, $e^{-i\varphi} = \cos\varphi - i\sin\varphi$,

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

Proof:

We use the Maclaurin series for the exponential function e^x, setting $x = i\varphi$:

$$e^{i\varphi} = 1 + \frac{i\varphi}{1!} - \frac{\varphi^2}{2!} - \frac{i\varphi^3}{3!} + \frac{\varphi^4}{4!} + \frac{i\varphi^5}{5!} + \dots$$

$$e^{-i\varphi} = 1 - \frac{i\varphi}{1!} - \frac{\varphi^2}{2!} + \frac{i\varphi^3}{3!} + \frac{\varphi^4}{4!} - \frac{i\varphi^5}{5!} + \dots$$

By adding and subtracting of these series we get:

$$e^{i\varphi} + e^{-i\varphi} = 2\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots\right) = 2\cos\varphi \quad \Rightarrow \quad \cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$e^{i\varphi} - e^{-i\varphi} = 2i\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots\right) = 2i\sin\varphi \quad \Rightarrow \quad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

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Derivatives utilization – L'Hospital's rule:

L'Hospital's rule for the evaluation of limits of indeterminate expressions (expressions of type 0/0 or ∞/∞).:

Description: L'Hospital's rule states that for functions *f* and *g* (which are differentiable on an interval *I*) it is valid:

$$\boxed{\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.}$$

when $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ is valid for all x in I with $x \neq c$.

Proof (very simple version - with Taylor series): Let us express functions *f* and *g* in a simplified version (using linearization):

$$f(c+h)\approx f(c)+hf'(c), \quad \mathrm{g}(c+h)\approx g(c)+hg'(c)$$

$$\lim_{h\to 0}\frac{f(c+h)}{g(c+h)}=\lim_{h\to 0}\frac{f(c)+hf'(c)}{g(c)+hg'(c)}=\lim_{h\to 0}\frac{hf'(c)}{hg'(c)}=\underline{\lim_{h\to 0}\frac{f'(c)}{g'(c)}}$$

provided that $f(c) = 0$ and $g(c) = 0$ (indeterminate expression 0/0).

(using functions $1/f$ and $1/g$ we can proof indeterminate expressions ∞/ ∞).

Derivatives utilization – L'Hospital's rule:

Examples:

1. Function $(5x–2)/(7x+3)$ for $x\to\infty$:

Find $\lim\limits_{x\to\infty} \dfrac{5x-2}{7x+3}$.

Solution 1: We have

$$\lim_{x\to\infty} \frac{5x-2}{7x+3} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(5x-2)'}{(7x+3)'} = \lim_{x\to\infty} \frac{(5x)'-2'}{(7x)'+3'} = \lim_{x\to\infty} \frac{5x'-2'}{7x'+3'}$$

$$= \lim_{x\to\infty} \frac{5\cdot 1-0}{7\cdot 1+0} = \lim_{x\to\infty} \frac{5}{7} = \frac{5}{7}$$

Solution 2: We have

$$\lim_{x\to\infty} \frac{5x-2}{7x+3} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\frac{5x-2}{x}}{\frac{7x+3}{x}} = \lim_{x\to\infty} \frac{\frac{5x}{x}-\frac{2}{x}}{\frac{7x}{x}+\frac{3}{x}} = \lim_{x\to\infty} \frac{5-\frac{2}{x}}{7+\frac{3}{x}} = \frac{5-0}{7+0} = \frac{5}{7}$$

<u>Comment:</u> Solution nr. 2 can be realized only in the case of rational functions.

Derivatives utilization – L'Hospital's rule:

Examples:

2. Function sin(x)/x for x→0 (example from lecture nr. 4, slide 4):

Find $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Solution: We have

$$\lim_{x\to 0}\frac{\sin x}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(\sin x)'}{x'}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{\cos 0}{1}=\frac{1}{1}=1$$

(here we can not use the approach from solution nr. 2 from previous slide)

3. Function ln(x)/√x for x→∞:

Find $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}}$.

Solution: We have

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln x)'}{(x^{1/2})'}=\lim_{x\to\infty}\frac{x^{-1}}{\frac{1}{2}x^{-1/2}}=\lim_{x\to\infty}\frac{x^{-1}\cdot x}{\frac{1}{2}x^{-1/2}\cdot x}=\lim_{x\to\infty}\frac{1}{\frac{1}{2}x^{1/2}}=0$$

Derivatives utilization – L'Hospital's rule:

Examples:

Back to previous lecture – proof of the derivative of sin(x) (slide nr. 26):

Trigonometric functions: *sin*(*x*) 2/2

Hence by the formulas

$$\boxed{\lim_{h\to 0}\frac{\sin(h)}{h}=1} \quad \text{and} \quad \boxed{\lim_{h\to 0}\frac{\cos(h)-1}{h}=0}$$

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}+\sin(x)\frac{\cos(h)-1}{h}\\ &= \cos(x)\cdot 1+\sin(x)\cdot 0\\ &= \cos(x).\end{aligned}$$

A similar computation leads to the derivative of $\cos x$.

Derivatives utilization – L'Hospital's rule:

When the result of L’Hospital’s rule application is still an indeterminate expressions of type 0/0 or ∞/∞, it can be applied again (and again...) – several times, until we get a determinate expression.

Example:

$$\lim_{x\to 0}\frac{5x-\sin 5x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(5x-\sin 5x)'}{(x^3)'}=\lim_{x\to 0}\frac{5-5\cos 5x}{3x^2}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{(5-5\cos 5x)'}{(3x^2)'}=\lim_{x\to 0}\frac{25\sin 5x}{6x}=\frac{125}{6}$$

Derivatives utilization – L'Hospital's rule:

L’Hospital’s rule can be applied to expressions of type 0·∞,

by means of reformulating them to 0/0 or ∞/∞.

Example:

$$\lim_{x\to 0+}(\sin x \ln x) = \lim_{x\to 0+}\frac{\ln x}{(\sin x)^{-1}} = \lim_{x\to 0+}\frac{(\ln x)'}{((\sin x)^{-1})'} = \lim_{x\to 0+}\frac{x^{-1}}{-(\sin x)^{-2}\cos x}$$

$$= -\lim_{x\to 0+}\frac{\sin^2 x}{x\cos x} = -\lim_{x\to 0+}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x}\right) = -1\cdot 0 = 0$$

Derivatives utilization – L'Hospital's rule:

Sometimes we have to solve quite special situations, like expressions of type: 0^0 or ∞^0 or 1^∞ (also indeterminate forms). Also here the L'Hospital's rule can be applied very effectively.

Example:

Find $\lim_{x\to\infty} x^{1/x}$.

Solution: Note that $\lim_{x\to\infty} x^{1/x}$ is ∞^0 type of an indeterminate form.

Put

$$y = x^{1/x}$$

then

$$\ln y = \ln x^{1/x} = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

We have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(\ln x)'}{x'} = \lim_{x\to\infty} \frac{x^{-1}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Therefore $$\lim_{x\to\infty} x^{1/x} = e^0 = 1$$

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Derivatives utilization – functions graph course analysis

Derivatives represent an important tool in the course analysis of functions graphs – the signs of the 1st and 2nd derivatives of a function tell us something about the shape of its graph.

- it is important to find parts of the function, which are increasing and decreasing (based on the sign of the first derivative),
- furthermore find the so called stationary points (f '(x) = 0), which are connected with extremes of the function,
- and finally distinguish, if these stationary points are extremes (maxima or minima) or inflection points – based on the sign of the second derivative of the function.

comment: very useful is also to find the zero points (points, where the function graph is crossing the x-axis)

Derivatives utilization – functions graph course analysis

Repetition form the 3rd lecture: increasing and decreasing functions – for all numbers *a* and *b* in the domain of the function *f* it is valid:

f is ...	if for all a and b one has...
Increasing:	$a < b \implies f(a) < f(b)$
Decreasing:	$a < b \implies f(a) > f(b)$
Non-increasing:	$a < b \implies f(a) \geq f(b)$
Non-decreasing:	$a < b \implies f(a) \leq f(b)$

Connection between these properties and sign of the first derivative gives the following theorem:

Theorem.

If a function is non-decreasing on an interval $a < x < b$ then $f'(x) \geq 0$ for all x in that interval.

If a function is non-increasing on an interval $a < x < b$ then $f'(x) \leq 0$ for all x in that interval.

This theorem follows from the sign of the expression, occurring in the limit of the derivative definition:

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Derivatives utilization – functions graph course analysis

increasing and decreasing function – example (cubic function):

Graph of a cubic function. Consider the function

$$y = f(x) = x^3 - x.$$

Its derivative is

$$f'(x) = 3x^2 - 1.$$

We try to find out where f' is positive, and where it is negative

$$f'(x) = 3\left(x^2 - \tfrac{1}{3}\right) = 3\left(x + \tfrac{1}{3}\sqrt{3}\right)\left(x - \tfrac{1}{3}\sqrt{3}\right)$$

from which you see that

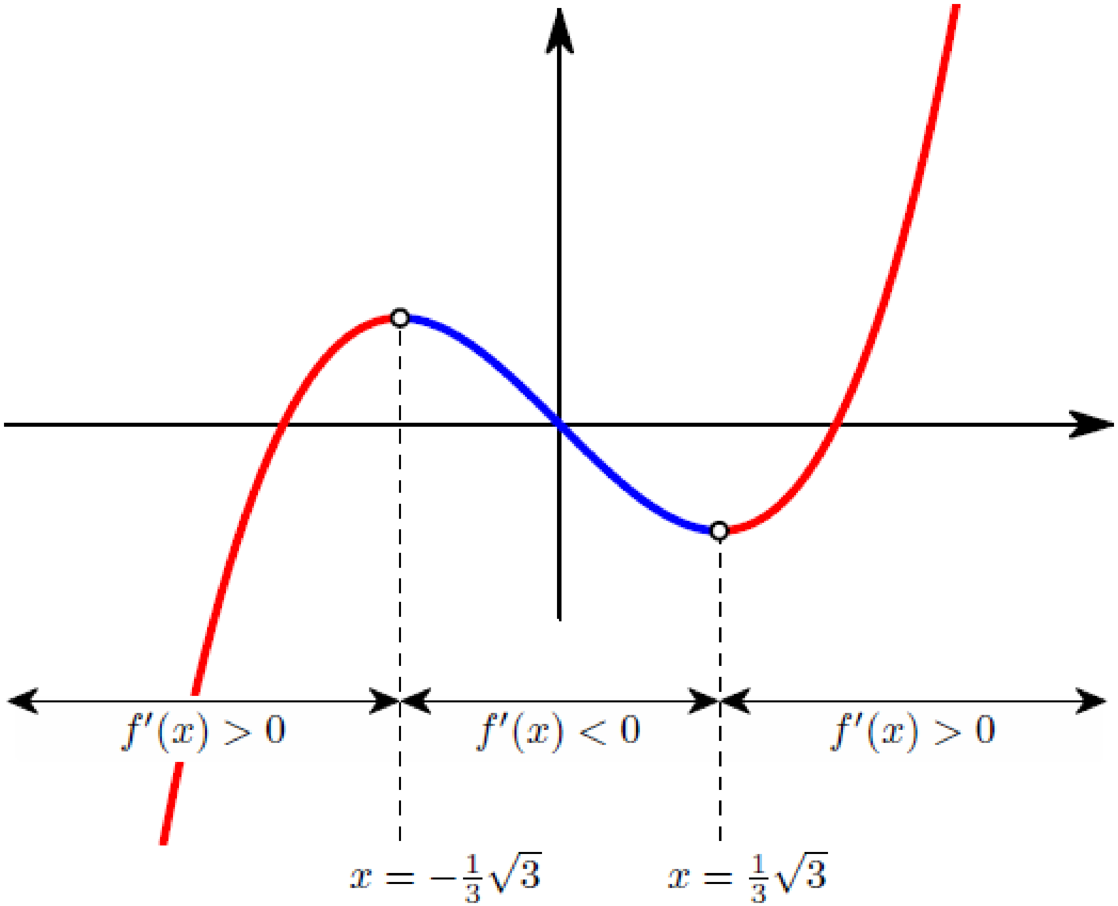
$$\begin{aligned} f'(x) &> 0 \text{ for } x < -\tfrac{1}{3}\sqrt{3} \\ f'(x) &< 0 \text{ for } -\tfrac{1}{3}\sqrt{3} < x < \tfrac{1}{3}\sqrt{3} \\ f'(x) &> 0 \text{ for } x > \tfrac{1}{3}\sqrt{3} \end{aligned}$$

Therefore the function f is

increasing on $\left(-\infty, -\tfrac{1}{3}\sqrt{3}\right)$, decreasing on $\left(-\tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3}\right)$, increasing on $\left(\tfrac{1}{3}\sqrt{3}, \infty\right)$.

Derivatives utilization – functions graph course analysis

increasing and decreasing function – example (cubic function):



The graph of $f(x) = x^3 - x$.

Therefore the function f is

increasing on $(-\infty, -\frac{1}{3}\sqrt{3})$, decreasing on $(-\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3})$, increasing on $(\frac{1}{3}\sqrt{3}, \infty)$.

Derivatives utilization – functions graph course analysis

Repetition form the 3rd lecture: maxima and minima of a function:

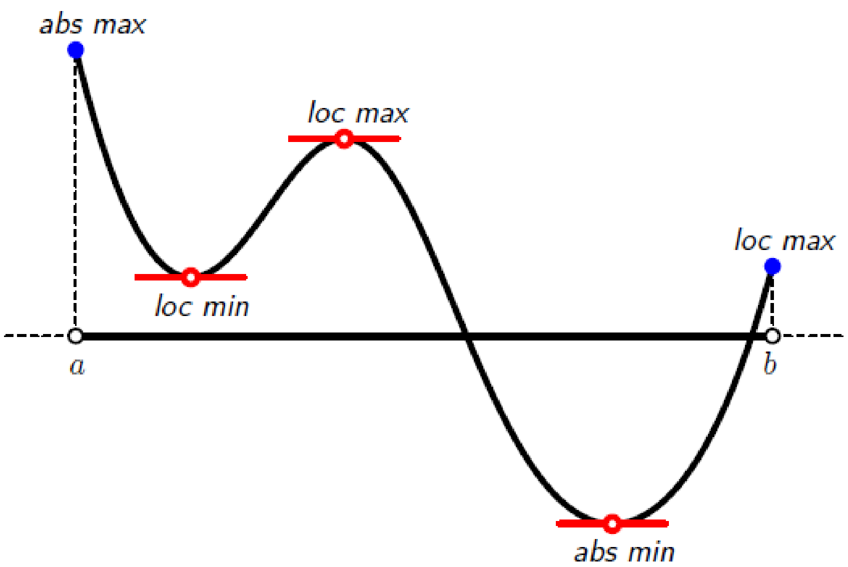
Maxima and Minima

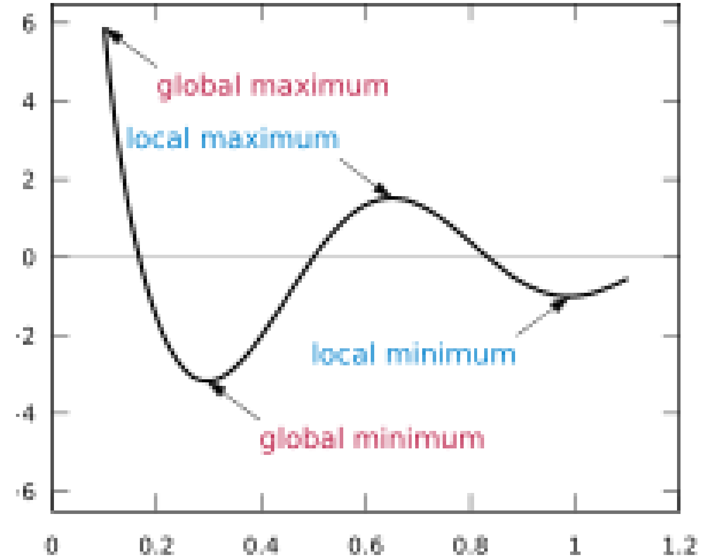
A function has a ***global maximum*** at some a in its domain if $f(x) \leq f(a)$ for all other x in the domain of f. Global maxima are sometimes also called "absolute maxima."

A function has a ***local maximum*** at some a in its domain if there is a small $\delta > 0$ such that $f(x) \leq f(a)$ for all x with $a - \delta < x < a + \delta$ which lie in the domain of f.

Every global maximum is a local maximum, but a local maximum doesn't have to be a global maximum.

Similar properties have also global minima and local minima.





Comment: maxima and minima - together are called extremes.

Maxima and minima of a function - connection to derivatives:

7.1. Where to find local maxima and minima. Any x value for which $f'(x) = 0$ is called a *stationary point* for the function f.

7.2. Theorem. *Suppose f is a differentiable function on some interval $[a, b]$.*

Every local maximum or minimum of f is either one of the end points of the interval $[a, b]$, or else it is a stationary point for the function f.

PROOF. Suppose that f has a local maximum at x and suppose that x is not a or b. By assumption the left and right hand limits

$$f'(x) = \lim_{\Delta x \nearrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ and } f'(x) = \lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

both exist and they are equal.

Since f has a local maximum at x we have $f(x + \Delta x) - f(x) \leq 0$ if $-\delta < \Delta x < \delta$. In the first limit we also have $\Delta x < 0$, so that

$$\lim_{\Delta x \nearrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \leq 0$$

Hence $f'(x) \leq 0$.

In the second limit we have $\Delta x > 0$, so

$$\lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0$$

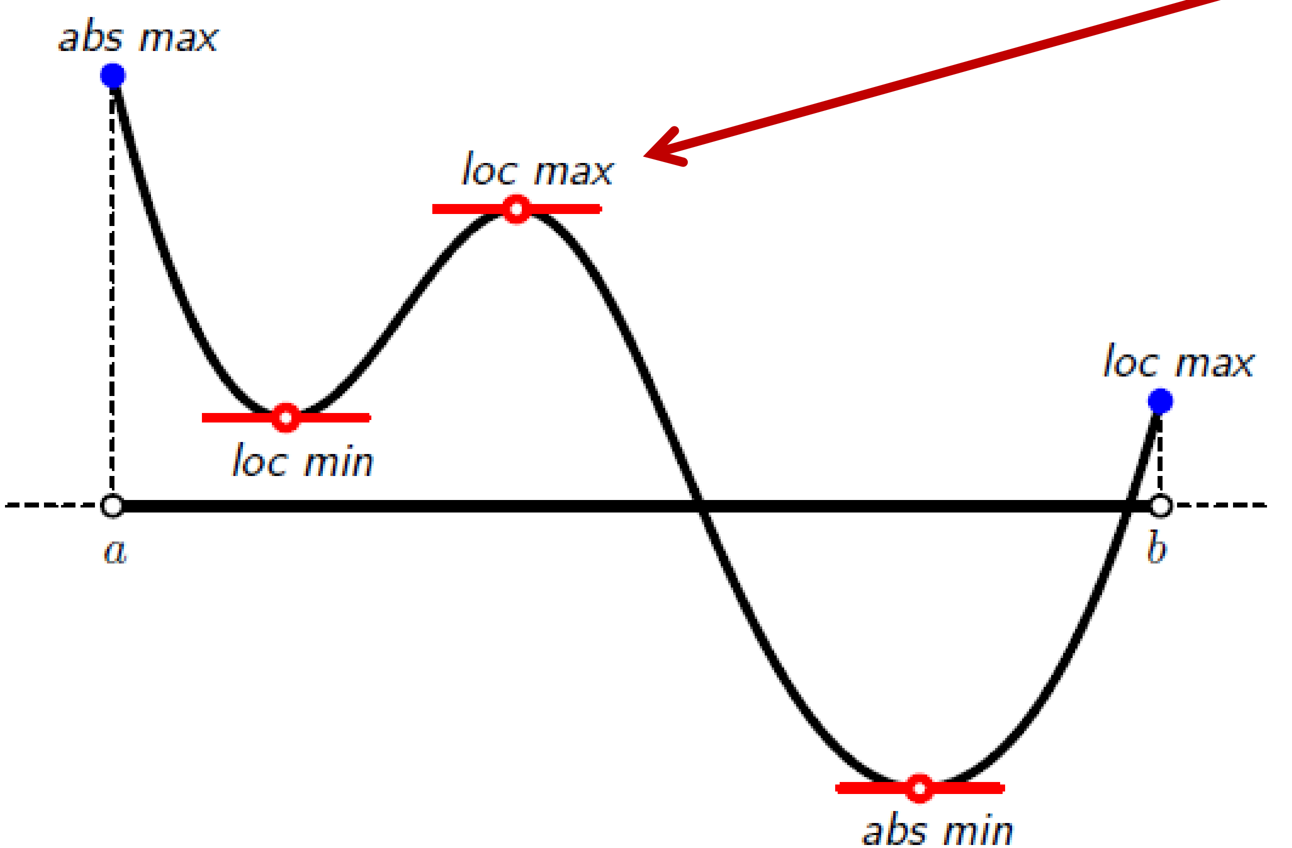
which implies $f'(x) \geq 0$.

Thus we have shown that $f'(x) \leq 0$ and $f'(x) \geq 0$ at the same time. This can only be true if $f'(x) = 0$.

Derivatives utilization – functions graph course analysis

Maxima and minima of a function - connection to derivatives:

In other words – an extreme of a function is defined by the condition f ’(x) = 0. This is used during the analysis of function courses (shapes of graphs).

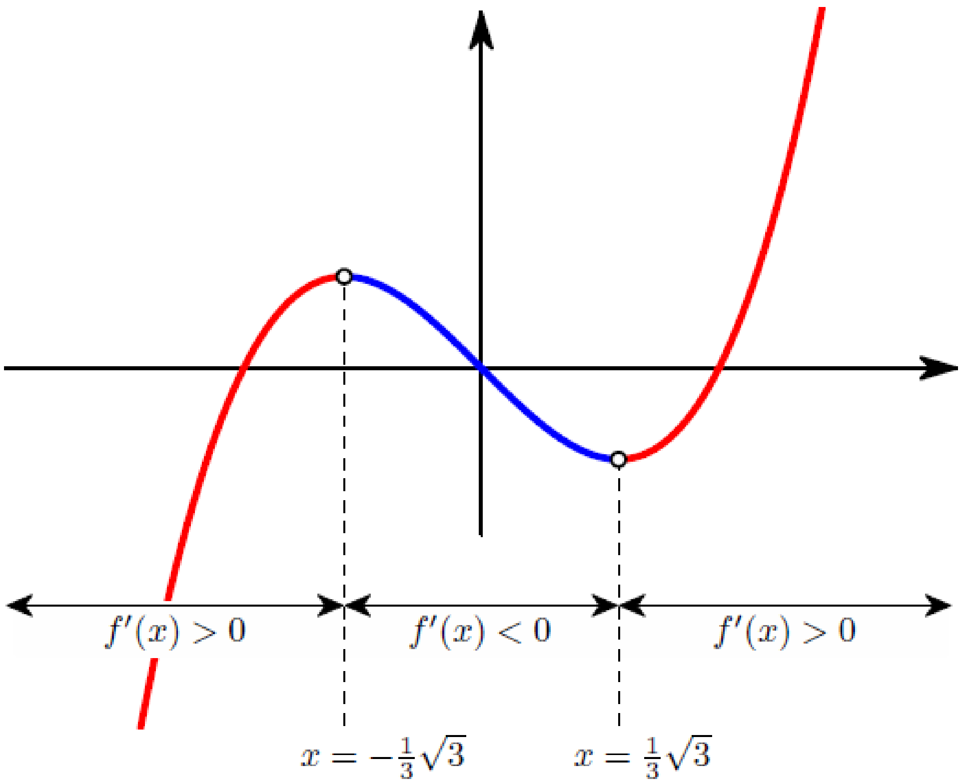


Slope of the tangent in such a graph point is equal to zero.

Beside this, first derivative change its sign in the close vicinity of these points

Maxima and minima of a function - connection to derivatives:

Example – local maxima and minima of $f(x) = x^3 - x$, we had found that the function $f(x) = x^3 - x$ is decreasing when $-\infty < x < -\frac{1}{3}\sqrt{3}$, and also when $\frac{1}{3}\sqrt{3} < x < \infty$, while it is increasing when $-\frac{1}{3}\sqrt{3} < x < \frac{1}{3}\sqrt{3}$. It follows that the function has a local minimum at $x = -\frac{1}{3}\sqrt{3}$, and a local maximum at $x = \frac{1}{3}\sqrt{3}$.



The graph of $f(x) = x^3 - x$.

Decision about minima/maxima was made on the increasing and decreasing property – but we have another tool for it, the second derivative.

Its derivative is $f'(x) = 3x^2 - 1$. (lets have a look back to slide nr. 24)

Derivatives utilization – functions graph course analysis

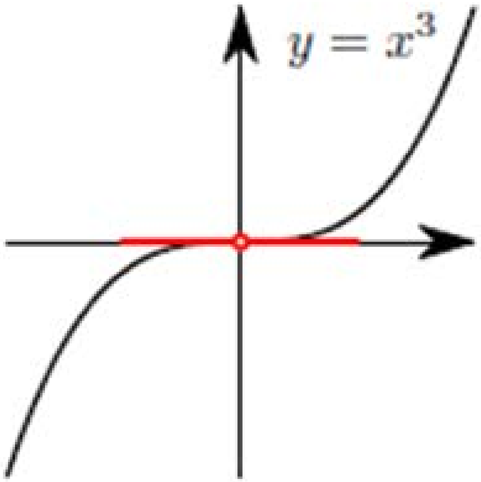
Maxima and minima of a function - connection to derivatives:

A stationary point that is neither a maximum nor a minimum.

If you look for stationary points of the function $f(x) = x^3$ you find that there's only one, namely $x = 0$.

The derivative $f'(x) = 3x^2$ does not change sign at $x = 0$.

And in fact, $x = 0$ is neither a local maximum nor a local minimum since $f(x) < f(0)$ for $x < 0$ and $f(x) > 0$ for $x > 0$.



This is so called inflection point (flex):

a point on a curve at which the curve changes from being concave to convex, or vice versa (we will come to it later).

Derivatives utilization – functions graph course analysis

Maxima and minima of a function - connection to derivatives:

When we find points, where f'(x) = 0, then these can be extremes (minima, maxima), but in some special cases also not.

Decision on this point we get, when we analyze the second derivative f''(x):

The second derivative test. We saw how you can tell if a stationary point is a local maximum or minimum by looking at the sign changes of $f'(x)$.

There is another way of distinguishing between local maxima and minima which involves computing the second derivative.

Theorem.

If c is a stationary point for a function f, and if $f''(c) < 0$ then f has a local maximum at $x = c$.

If $f''(c) > 0$ then f has a local minimum at c.

The theorem doesn't say what happens when $f''(c) = 0$.

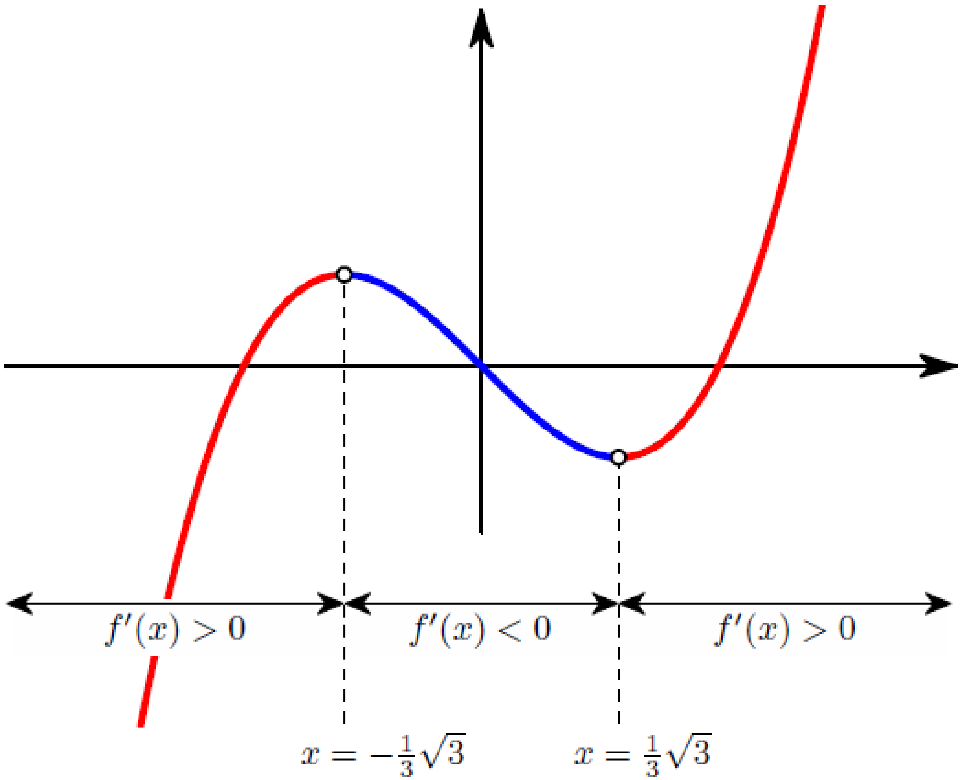
In that case you must go back to checking the signs of the first derivative near the stationary point.

Maxima and minima of a function - connection to derivatives:

Example – that cubic function again. Consider the function $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$. Since $f''(x) = 6x$ we have

$$f''\left(-\tfrac{1}{3}\sqrt{3}\right) = -2\sqrt{3} < 0 \text{ and } f''\left(\tfrac{1}{3}\sqrt{3}\right) = 2\sqrt{3} > 0.$$

Therefore f has a local maximum at $-\frac{1}{3}\sqrt{3}$ and a local minimum at $\frac{1}{3}\sqrt{3}$.



The graph of $f(x) = x^3 - x$.

Derivatives utilization – functions graph course analysis

Convex and concave shapes (inflection points):

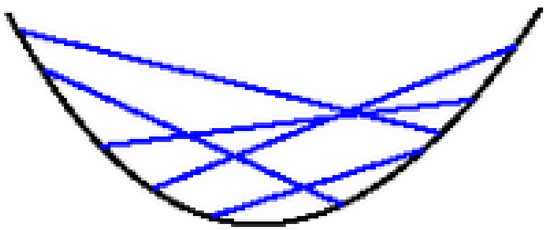
By definition, a function f is ***convex*** on some interval $a < x < b$ if the line segment connecting any pair of points on the graph lies *above* the piece of the graph between those two points.

The function is called ***concave*** if the line segment connecting any pair of points on the graph lies *below* the piece of the graph between those two points.

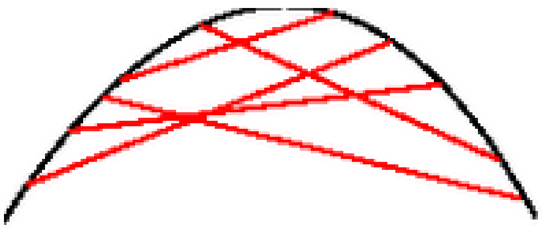
A point on the graph of f where $f''(x)$ changes sign is called an ***inflection point***.

Instead of "convex" and "concave" one often says "curved upwards" or "curved downwards."

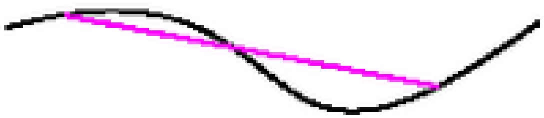
You can use the second derivative to tell if a function is concave or convex.



A convex function.
No line segment lies below the graph at any point.



A concave function.
No line segment lies above the graph at any point.

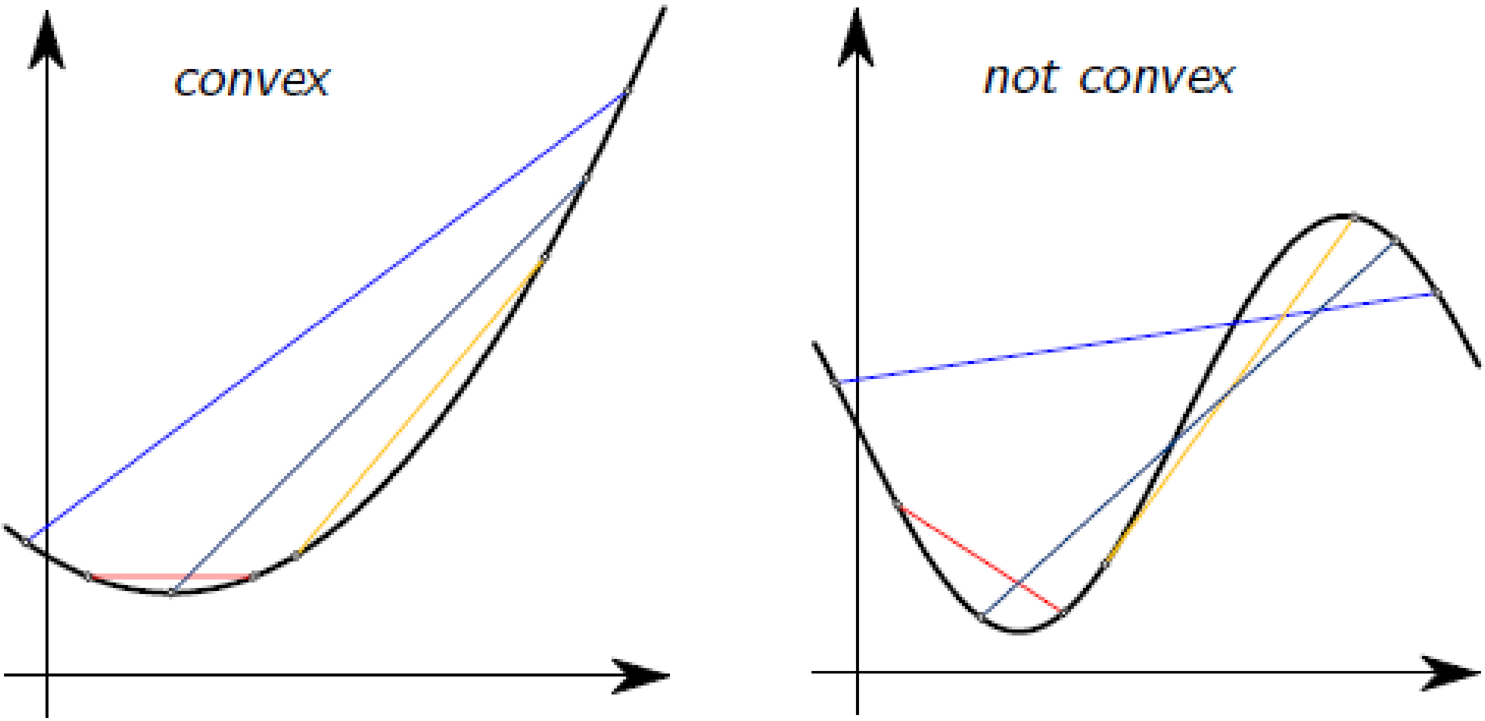


A function that is neither concave nor convex.

Comment: There are quite different modifications of these terms – we will come to it later on.

Derivatives utilization – functions graph course analysis

Convex and concave shapes (inflection points):



If a graph is convex then all chords lie above the graph.
If it is not convex then some chords will cross the graph or lie below it.

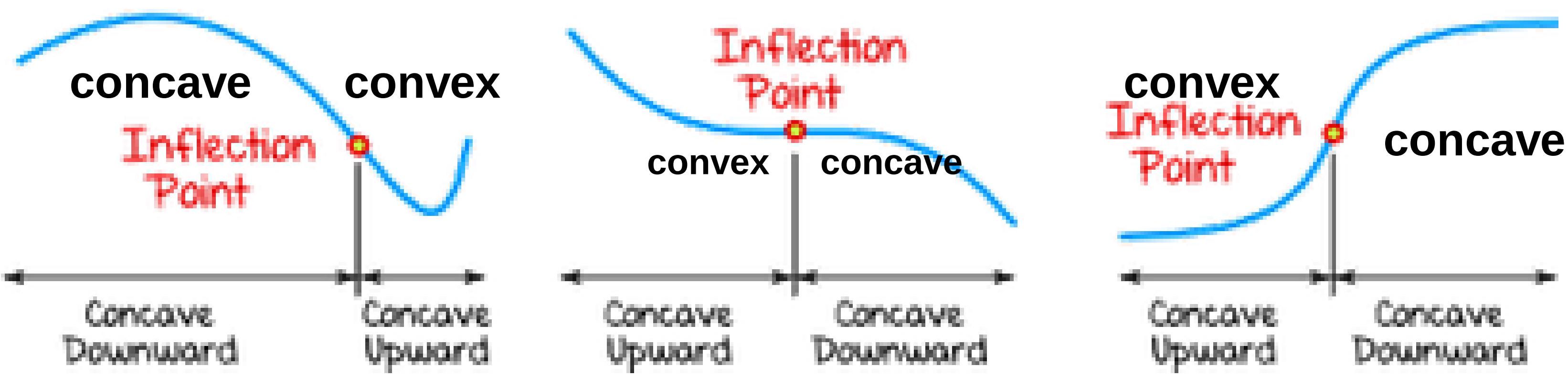
Quite good visualization of transition between convex and concave:
https://en.wikipedia.org/wiki/Inflection_point

Convex and concave shapes:

Different terms are used worldwide:

Convex function is also synonymously called convex downward or concave upward.

Concave function is also synonymously called concave downward or convex upward.



example from www.mathsisfun.com.

Comment: In some scientific branches (e.g. cartography, geomorphology) these two terms are switched.

Example – functions graph course analysis (finding extremes):

Example: Find the maxima and minima for:

$$y = 5x^3 + 2x^2 - 3x$$

The derivative (slope) is:

$$\frac{d}{dx} y = 15x^2 + 4x - 3$$

Which is quadratic with zeros at:

- $x = -3/5$
- $x = +1/3$

Could they be maxima or minima? (Don't look at the graph yet!)

The second derivative is $\mathbf{y'' = 30x + 4}$

At $x = -3/5$:

$y'' = 30(-3/5) + 4 = -14$

it is less than 0, so $-3/5$ is a local maximum

Example – functions graph course analysis (finding extremes):

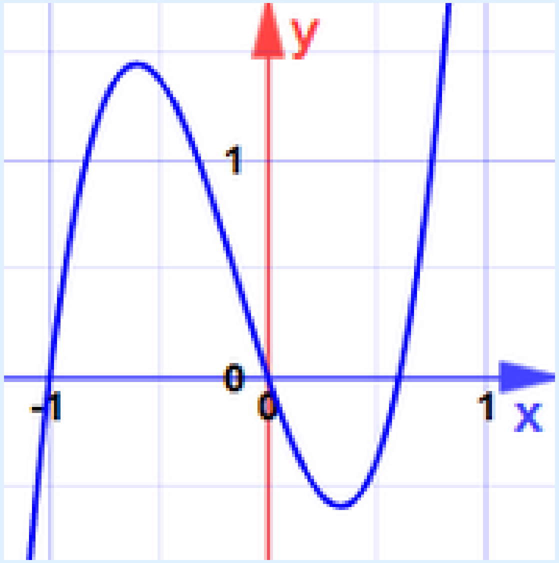
The second derivative is **y'' = 30x + 4**

At x = +1/3:

→ $y'' = 30(+1/3) + 4 = +14$

→ it is greater than 0, so +1/3 is a local minimum

(Now you can look at the graph.)



For the exact graph construction we need also the zero points (obtained from the solved equation y = 0).

Next example – functions graph course analysis (finding extremes):

Example: Find the maxima and minima for:

$$y = x^3 - 6x^2 + 12x - 5$$

The derivative is:

$$\frac{d}{dx}y = 3x^2 - 12x + 12$$

Which is quadratic with only one zero at $\mathbf{x = 2}$

Is it a maximum or minimum?

Next example – functions graph course analysis (finding extremes):

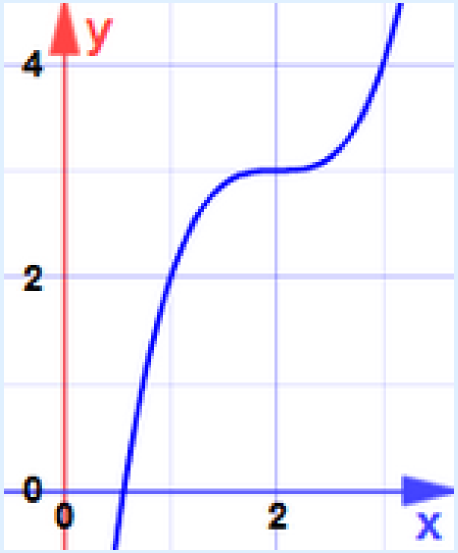
The second derivative is **y'' = 6x − 12**

At x = 2:

$y'' = 6(2) - 12 = 0$

it is 0, so the test fails

And here is why:



It is a **saddle point** ... the slope does become zero, but it is neither a maximum or minimum.

Comment: Saddle point – it is an inflection point of the function.

Lecture 6: differentiation – further topics, complex numbers

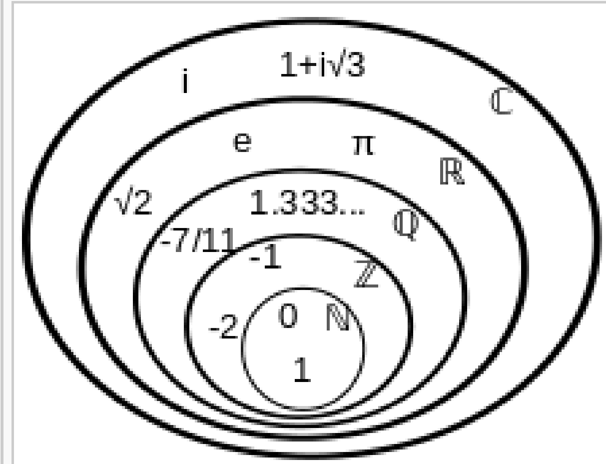
Content:

- derivatives utilization - Taylor series
- derivatives utilization - L'Hospital's rule
- functions graph course analysis

- complex numbers, introduction
- complex numbers, basic operations
- complex numbers, functions

Slide from 1. lecture: Basic objects in mathematics: **numbers, sets**

Different types of numbers have many different uses.
Numbers can be classified into sets, called **number systems**.



Subsets of the complex numbers.

Main number systems

$\mathbb{N}$	**Natural**	0, 1, 2, 3, 4, ... **or** 1, 2, 3, 4, ...
$\mathbb{Z}$	**Integer**	..., −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, ...
$\mathbb{Q}$	**Rational**	$\frac{a}{b}$ where *a* and *b* are integers and *b* is not 0
$\mathbb{R}$	**Real**	The limit of a convergent sequence of rational numbers
$\mathbb{C}$	**Complex**	*a* + *bi* where *a* and *b* are real numbers and *i* is the square root of −1

$1.5 = \frac{3}{2}$ Ratio

Rational

$\pi = 3.14159... = \frac{?}{?}$ (No Ratio)

Irrational

i – so called imaginary unit

Complex numbers - introduction

Definition: A complex number is a number that can be expressed in the form $a + ib$, where a and b are real numbers and i is the imaginary unit, that satisfies the equation $i^2 = -1$.

a – the so called real part of the complex number,

b – the so called imaginary part of the complex number.

Complex numbers extend the concept of the one-dimensional number line to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part.

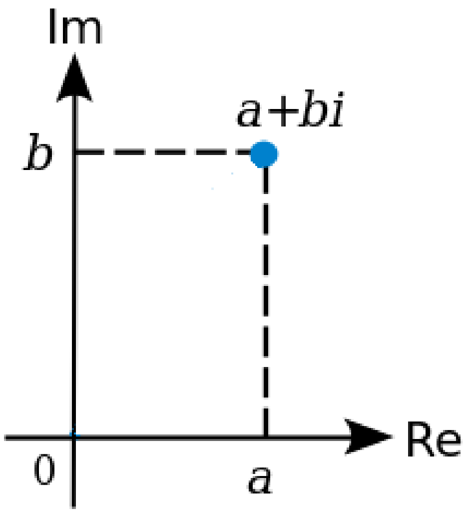
The complex number $a + ib$ can be identified with the point (a, b) in the complex plane.

A complex number whose real part is zero is said to be purely imaginary ($0+ib$), whereas a complex number whose imaginary part is zero is a real number ($a+i0$). In this way, the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone.

Example:

$$\mathrm{Re}(-3.5 + 2i) = -3.5$$

$$\mathrm{Im}(-3.5 + 2i) = 2$$



Complex numbers - introduction

Definition: A complex number is a number that can be expressed in the form *a* + *ib*, where *a* and *b* are real numbers and *i* is the imaginary unit, that satisfies the equation $i^2 = -1$.

a – the so called real part of the complex number,

b – the so called imaginary part of the complex number.

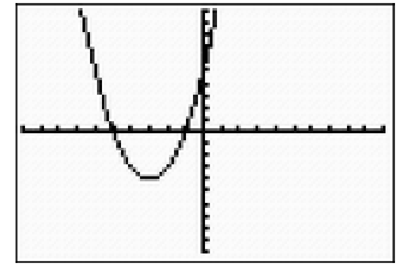
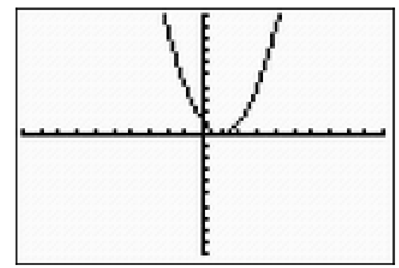
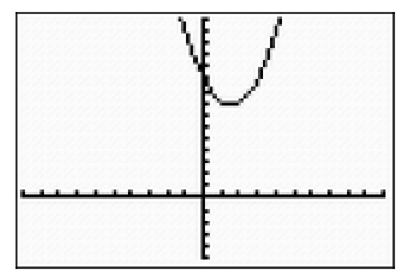
Imaginary unit *i* was not selected randomly as $i^2 = -1$, but because this plays very important role in various solutions of equations and other mathematical problems.

Example (quadratic equation with negative discriminant):

$$x^2 - 3x + 10 = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 10}}{2} = \frac{3 \pm \sqrt{-31}}{2} = \frac{3 \pm i\sqrt{31}}{2} = \begin{array}{l} x_1 = 3/2 + (i/2)\sqrt{31} \\ x_2 = 3/2 - (i/2)\sqrt{31} \end{array}$$

geometrical presentation of quadratic equation solutions:

Value of the discriminant	Example showing nature of roots of $ax^2 + bx + c = 0$	Graph indicating x-intercepts $y = ax^2 + bx + c$
POSITIVE $b^2 - 4ac > 0$	$x^2 + 6x + 5 = 0$ $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$ $x = \frac{-6 \pm \sqrt{16}}{2} = \frac{-6 \pm 4}{2}$ $x = -1; \quad x = -5$ **There are two real roots.** (If the discriminant is a perfect square, the two roots are rational numbers. If the discriminant is not a perfect square, the two roots are irrational numbers containing a radical.)	 **There are two x-intercepts.**
ZERO $b^2 - 4ac = 0$	$x^2 - 2x + 1 = 0$ $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$ $x = \frac{2 \pm \sqrt{0}}{2} = 1$ $x = 1; \quad x = 1$ **There is one real root.** (The root is repeated.)	 **There is one x-intercept.**
NEGATIVE $b^2 - 4ac < 0$	$x^2 - 3x + 10 = 0$ $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(10)}}{2(1)}$ $x = \frac{3 \pm \sqrt{-31}}{2}$ $x = \frac{3 + i\sqrt{31}}{2}; \quad x = \frac{3 - i\sqrt{31}}{2}$ **There are two complex roots.**	 **There are no x-intercepts.**

Complex numbers - introduction

Definition: A complex number is a number that can be expressed in the form *a* + *ib*, where *a* and *b* are real numbers and *i* is the imaginary unit, that satisfies the equation $i^2 = -1$.

a – the so called real part of the complex number,

b – the so called imaginary part of the complex number.

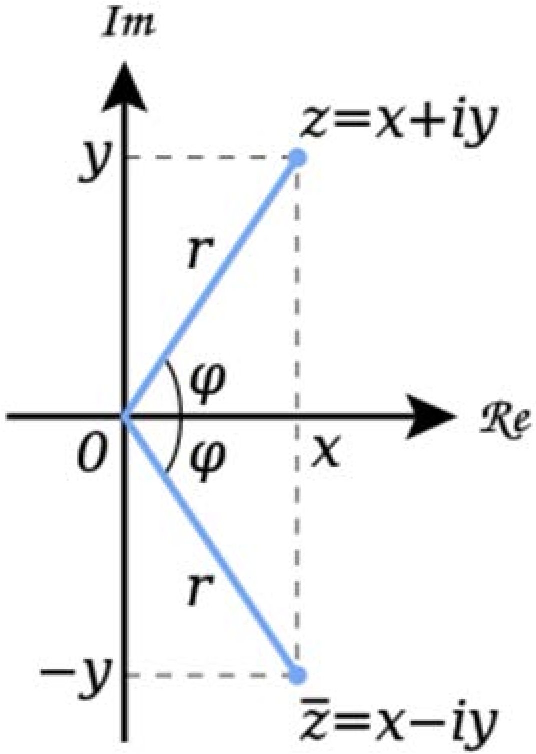
back to the previous example:

$$x_1 = 3/2 + (i/2)\sqrt{31}$$
$$x_2 = 3/2 - (i/2)\sqrt{31}$$

Complex solutions are plotted in the complex plane (so called Argand diagram):

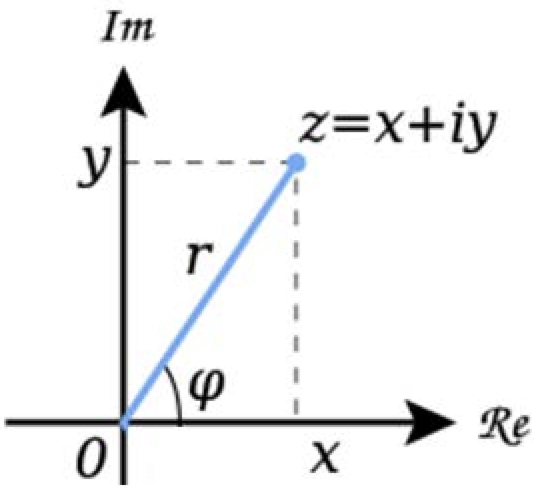
$z = a + ib = x + iy$

Comment: $\bar{z}$ is called as complex conjugate number (imaginary part has opposite sign).



Complex numbers - introduction

Form: A complex number can be written in two basic forms:



1. Cartesian form (with real and imaginary parts):

   $$z = a + ib = x + iy$$

2. goniometric or exponential form

   (using the so called Euler's formula):

   $$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}.$$

where:

r – modulus (magnitude) of a complex number

(also called as absolute value: $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$)

φ – argument of the complex number: $Arg(z) = \varphi = arctg(y/x)$

it is also valid (for k = 0, 1, …): $\arg(z) = \varphi = arctg(y/x) + 2\pi k$

Comment to the imaginary unit:

$i = \sqrt{-1}$, $i^1 = i$, $i^2 = -1$, $i^3 = i \cdot i^2 = -i$, $i^4 = i^2 \cdot i^2 = 1$, $i^5 = i^4 \cdot i = i$

$i^{-1} = i/i^2 = -i$, …

Complex numbers – introduction

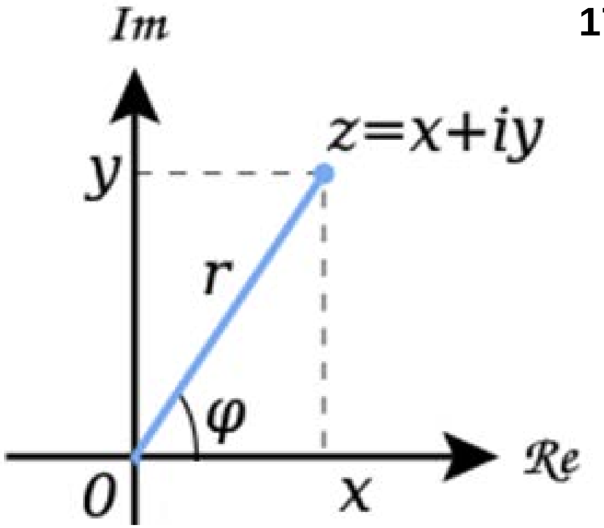
Leonhard Euler
1707 - 1783

Euler's formulas:

$e^{i\varphi} = \cos\varphi + i\sin\varphi$, $e^{-i\varphi} = \cos\varphi - i\sin\varphi$,

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$



Derivation (proof) of these formulas will be given later.

Complex numbers – basic operations

Equality:

Two complex numbers are equal if and only if both their real and imaginary parts are equal.

We can write this in math-symbols:

$$z_1 = z_2 \;\leftrightarrow\; (\mathrm{Re}(z_1) = \mathrm{Re}(z_2) \wedge \mathrm{Im}(z_1) = \mathrm{Im}(z_2))$$

Conjugation:

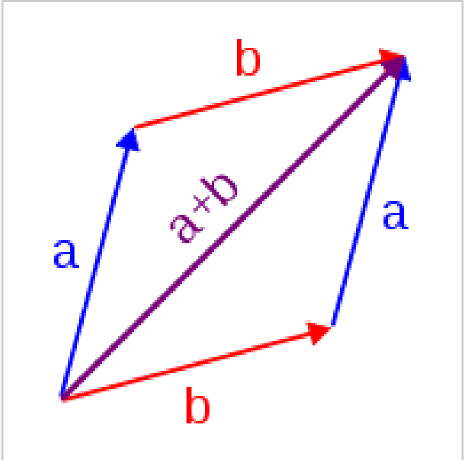
The complex conjugate of the complex number $z = x + iy$ is defined to be $x - iy$. It is denoted $\bar{z}$ or z^* .

Geometrically, conjugate complex number is the "reflection" of the complex number about the real axis. Conjugating twice gives the original complex number:

Conjugation distributes over the standard arithmetic operations:

$$\overline{z+w} = \bar{z} + \bar{w},$$
$$\overline{z-w} = \bar{z} - \bar{w},$$
$$\overline{zw} = \bar{z}\bar{w},$$
$$\overline{(z/w)} = \bar{z}/\bar{w}.$$

Complex numbers – basic operations



Addition of two complex numbers can be done geometrically by constructing a parallelogram.

Addition and subtraction:

Complex numbers are added by adding the real and imaginary parts of the summands.

Addition:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

Similarly, subtraction:

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Multiplication and division:

The multiplication of two complex numbers is defined by the following formula:

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

And, division:

$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

Complex numbers – basic operations

Multiplication and division:

We can use in a much more straight-forward way the exponential form of complex numbers:

$$z_1 z_2 = r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$$

Or in the goniometric form:

$$z_1 z_2 = r_1 r_2 \left[\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)\right]$$

Complex numbers – basic operations

Raising to a power (exponentiation):

Here we can express this kind of operation much more effectively again with the exponential form of complex numbers:

$$z^n = \left[re^{i\varphi}\right]^n = r^n e^{in\varphi}$$

In Cartesian form this operation is more complicated and it utilises the so called binomial rule:

$$z^n = (a+ib)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (ib)^k$$

$$\text{where} \quad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Complex numbers – basic operations

*n*th root evaluation:

Also here we can express this kind of operation much more effectively again with the exponential form of complex numbers:

$$\sqrt[n]{z} = z^{1/n} = \left[re^{i\varphi}\right]^{1/n} = \left[re^{i(\varphi+2k\pi)}\right]^{1/n} = r^{1/n}e^{i(\varphi+2k\pi)/n} =$$

$$= r^{1/n}e^{i(\varphi/n+2k\pi/n)} = \sqrt[n]{r}\left[\cos(\varphi/n + 2k\pi/n) + i\sin(\varphi/n + 2k\pi/n)\right]$$

Here we can see that the result of *n*th root of a complex number is not unique – it has *n* solutions (for *k* = 0, 1, 2, ... *n*–1).

These solutions lie on a circle with the radius $r^{1/n}$ and their arguments are different by $2\pi/n$.

$$\zeta_k = \sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\varphi/n + 2k\pi/n\right) + i\sin\left(\varphi/n + 2k\pi/n\right)\right]$$

***n*th root evaluation – example:**

We have a complex number $z = 8\sqrt{2} + i8\sqrt{2}$

and we have to evaluate its 4th root: $\sqrt[4]{z}$.

First of all we have to evaluate its modulus r and the main argument φ,

$r = [(8\sqrt{2})^2+(8\sqrt{2})^2]^{1/2} = 16$ and the main argument $\varphi = \text{arctg}(1) = \pi/4$.

Therefore we can write: $z = 16e^{i\pi/4} = 16[\cos(\pi/4)+i\sin(\pi/4)]$ and for $\sqrt[4]{z}$:

$$\zeta_k = \sqrt[4]{z} = \sqrt[4]{16}\left[\cos\left(\varphi/4 + 2k\pi/4\right) + i\sin\left(\varphi/4 + 2k\pi/4\right)\right] =$$

$$= 2\left[\cos\left(\pi/16 + k\pi/2\right) + i\sin\left(\pi/16 + k\pi/2\right)\right] \quad \text{for k} = 0, 1, 2, 3.$$

$$\zeta_0 = 2\left[\cos(\pi/16) + \mathrm{i}\sin(\pi/16)\right] = 2e^{\mathrm{i}(\pi/16)}$$

$$\zeta_1 = 2\left[\cos(\pi/16 + \pi/2) + \mathrm{i}\sin(\pi/16 + \pi/2)\right] = 2e^{\mathrm{i}(\pi/16+\pi/2)}$$

$$\zeta_2 = 2\left[\cos(\pi/16 + \pi) + \mathrm{i}\sin(\pi/16 + \pi)\right] = 2e^{\mathrm{i}(\pi/16+\pi)}$$

$$\zeta_3 = 2\left[\cos(\pi/16 + 3\pi/2) + \mathrm{i}\sin(\pi/16 + 3\pi/2)\right] = 2e^{\mathrm{i}(\pi/16+3\pi/2)}$$

$$\zeta_k = \sqrt[4]{z} = \sqrt[4]{16}\left[\cos\left(\varphi/4 + 2k\pi/4\right) + i\sin\left(\varphi/4 + 2k\pi/4\right)\right] =$$

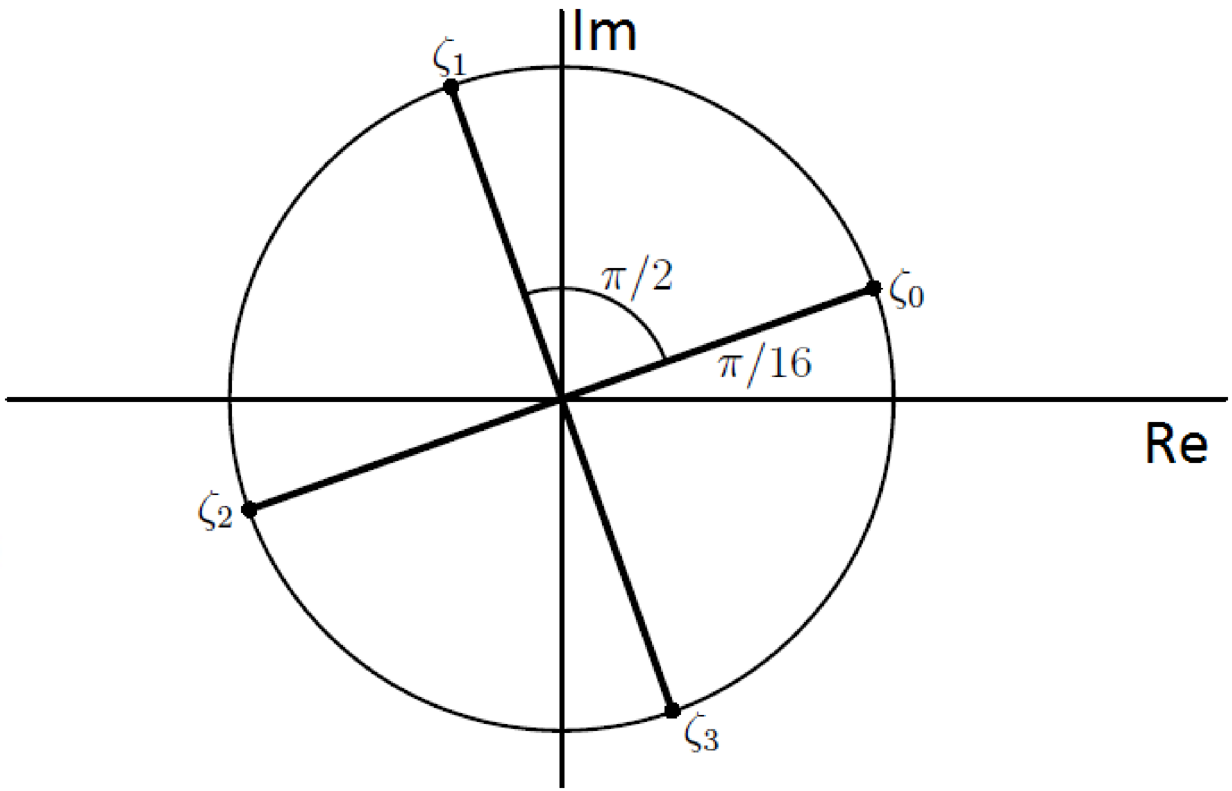
$$= 2\left[\cos\left(\pi/16 + k\pi/2\right) + i\sin\left(\pi/16 + k\pi/2\right)\right] \quad \text{for k} = 0, 1, 2, 3.$$

$$\zeta_0 = 2e^{\mathrm{i}(\pi/16)}$$

$$\zeta_1 = 2e^{\mathrm{i}(\pi/16+\pi/2)}$$

$$\zeta_2 = 2e^{\mathrm{i}(\pi/16+\pi)}$$

$$\zeta_3 = 2e^{\mathrm{i}(\pi/16+3\pi/2)}$$



These 4 solutions lie on a circle with the radius 2 and their arguments are different by $\pi/2$.

Complex numbers – functions

We distinguish here two basic types:

a) complex functions of real variable,

b) complex functions of complex variable.

A) Complex function of real variable:

we call $z = \mathrm{f}(t)$ as complex function of real variable, when $z \in \mathrm{C}$ and $t \in \mathrm{R}$.

Examples:

1. $z = t\,\mathrm{e}^{i\alpha}$, $t \in \langle 0, +\infty \rangle$, α = const.

Values of t from the interval $\langle 0, +\infty \rangle$ are displayed into a half-line in the complex plane (forming an angle α with the real axis).

$z = t\,\mathrm{e}^{i\alpha} = t(\cos\alpha + i\sin\alpha) \Rightarrow x = t\cos\alpha,\ y = t\sin\alpha \Rightarrow y = x\,\mathrm{tg}\,\alpha.$

(this is an equation for a line)

A) Complex function of real variable:

we call z = f(t) as complex function of real variable, when $z \in$ C and $t \in$ R

Examples:

2. $z = r\, e^{it}$, r = const., $t \in \langle\alpha, \beta\rangle$, $\alpha \geq 0$, $\beta \leq 2\pi$.

Values of t from the interval $\langle\alpha, \beta\rangle$ are displayed into a circle-arc in the complex plane (forming an angle α with the real axis).

$z = r\, e^{it} = r(\cos t + i \sin t) \Rightarrow x = r\cos t,\ y = r\sin t.$

When we build sum of squares $x^2 + y^2$, we obtain:

$x^2+y^2 = r^2\cos^2 t+r^2\sin^2 t = r^2 (\cos^2 t+\sin^2 t) = r^2 \Rightarrow x^2+y^2 = r^2,$

(this is the equation of a circle)

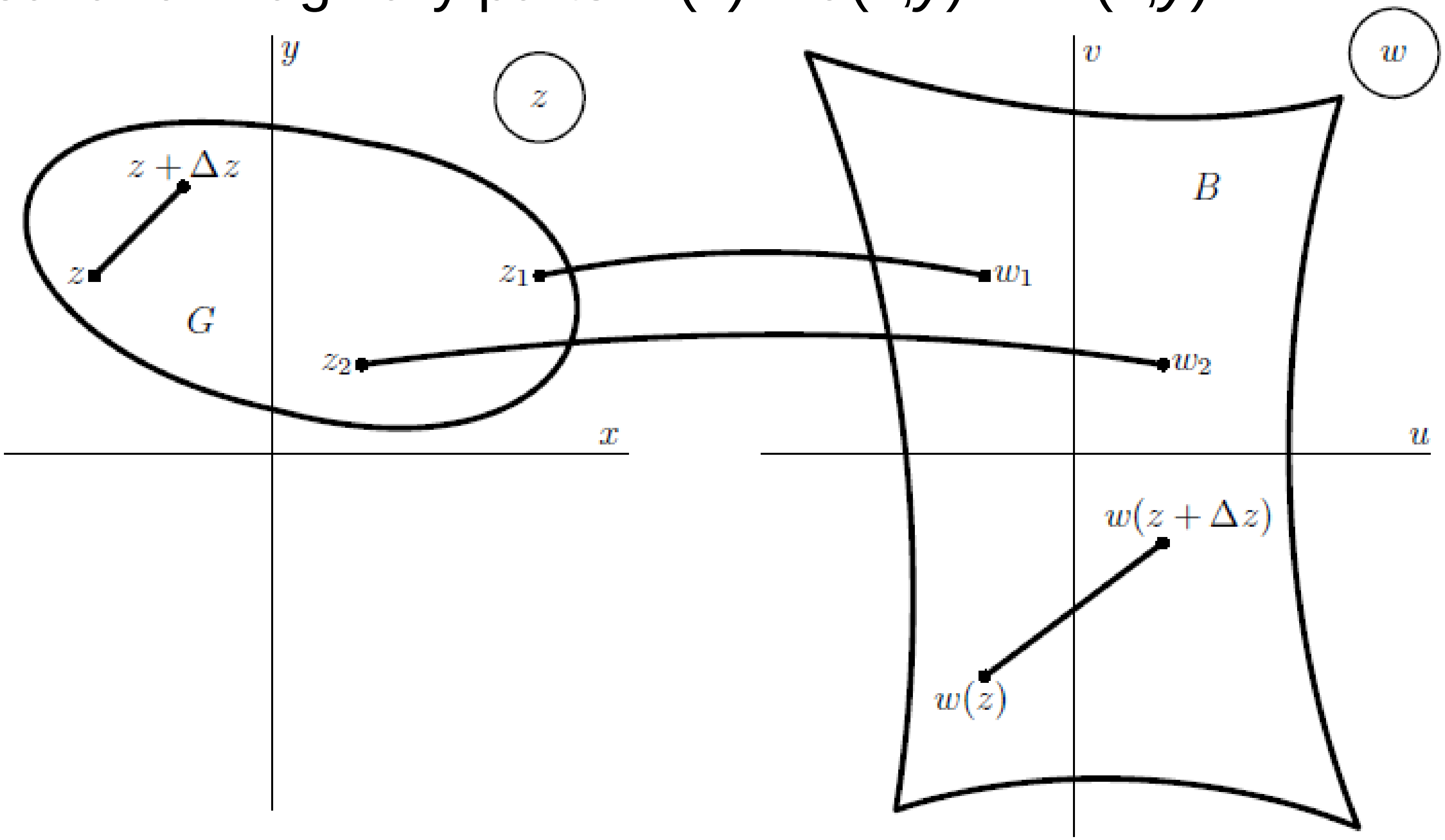
3. $z - z_0 = r\, e^{it}$, r = const., $t \in \langle 0, 2\pi\rangle$.

This is a very similar example, compared with the previous one, result is a circle in the complex plane with the centre in point z_0.

Complex numbers – functions

B) Complex function of complex variable:

we call $w = f(z)$ as complex function of complex variable, when $w \in C$ and also $z \in C$. Such a function can be divided into its real and imaginary parts: $f(z) = u(x,y) + i\,v(x,y)$.



These functions can have completely different properties like real number functions.

Complex numbers – functions

B) Complex function of complex variable:

Example: f(z) = cos(z) .

For its analysis we use one of Euler's formulas: $\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$

$$\cos(z) = \frac{1}{2}(e^{\mathrm{i}\,z} + e^{-\mathrm{i}\,z})$$

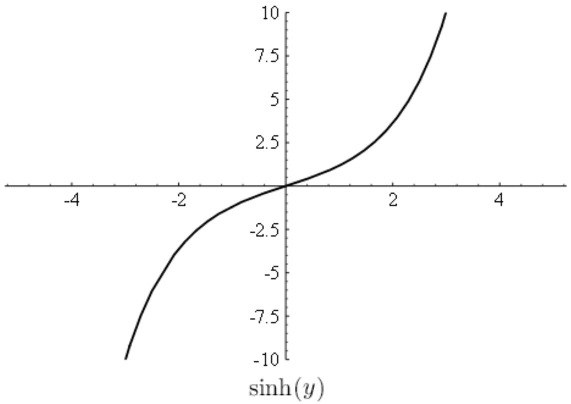
setting for $z = x + iy$ and by reformulating we get:

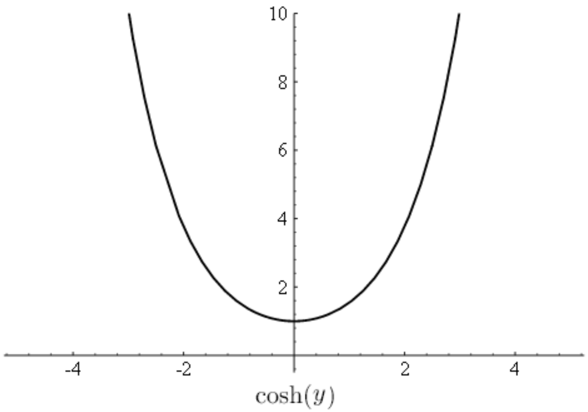
$$\cos(z) = \cos(x + \mathrm{i}\,y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

where we have used the hyperbolic functions sinh(x) and cosh(x):

$$\cosh(y) = \frac{1}{2}(e^{y} + e^{-y})$$

$$\sinh(y) = \frac{1}{2}(e^{y} - e^{-y})$$





B) Complex function of complex variable:

Example: f(z) = cos(z)

$$\cos(z) = \cos(x + \mathrm{i}\, y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

Recognising here the form of this kind of function: f(z) = u(x,y) + i v(x,y):

$$\mathrm{Re}[\cos(z)] = u(x, y) = \cos(x)\cosh(y)$$

$$\mathrm{Im}[\cos(z)] = v(x, y) = -\sin(x)\sinh(y)$$

First important thing, which we can see is that this function is not limited (like it was valid for the real function cos(x)), but it can grow to ±∞ due to the properties of hyperbolic sine and cosine:

$$|\cos(z)| = \left(u^2 + v^2\right)^{1/2} = \left\{[\cos(x)\cosh(y)]^2 + [\sin(x)\sinh(y)]^2\right\}^{1/2}$$

Next property is its periodicity:

cos(z) = cos(z + 2$k\pi$), when k = 1, 2, ...

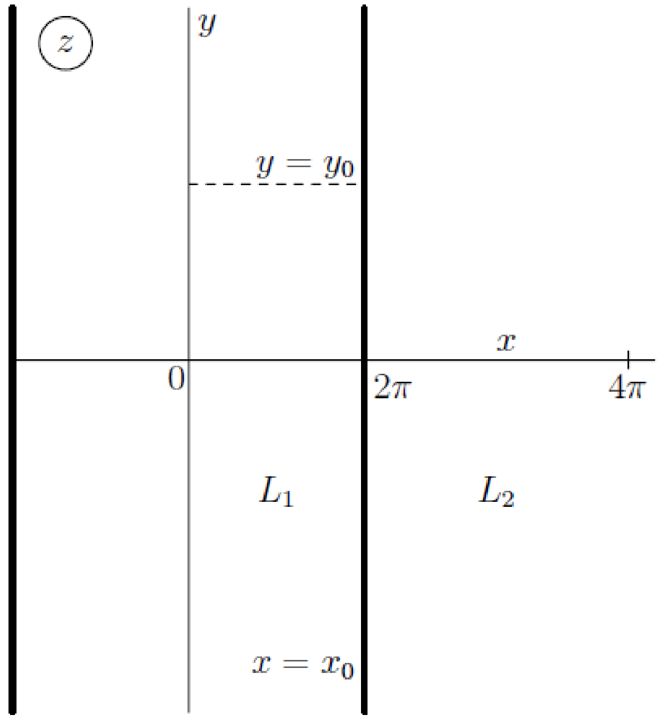
B) Complex function of complex variable:

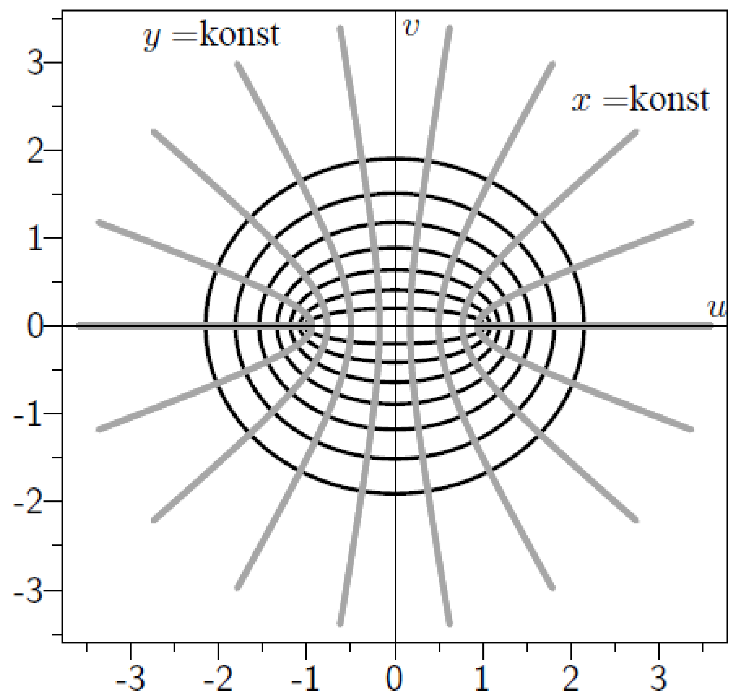
Example: f(z) = cos(z)

$$\cos(z) = \cos(x + \mathrm{i}\, y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

Next thing is the analysis of its geometrical properties:

1. First, we take a line segment $y = y_0$, $x \in \langle 0, 2\pi \rangle$





Using $\cos^2 x + \sin^2 x = 1$ we get:

$$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1$$

This is equation of an ellipse.

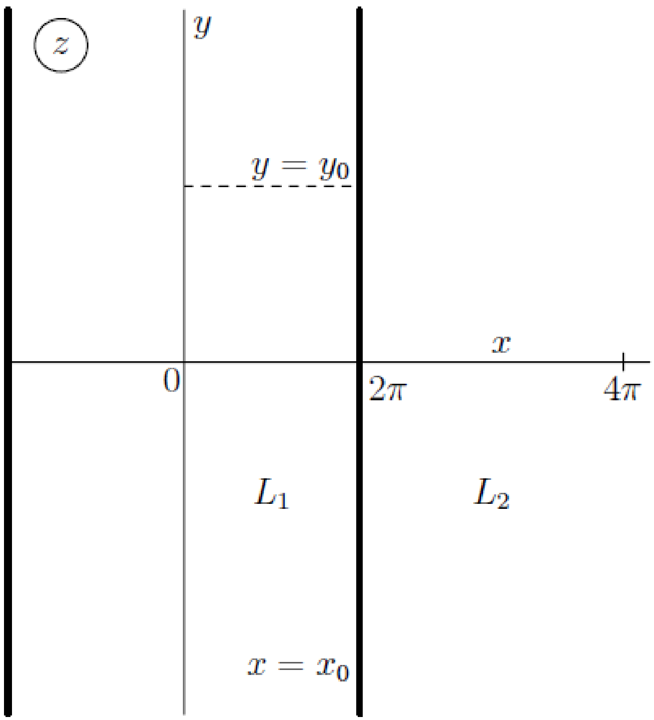
B) Complex function of complex variable:

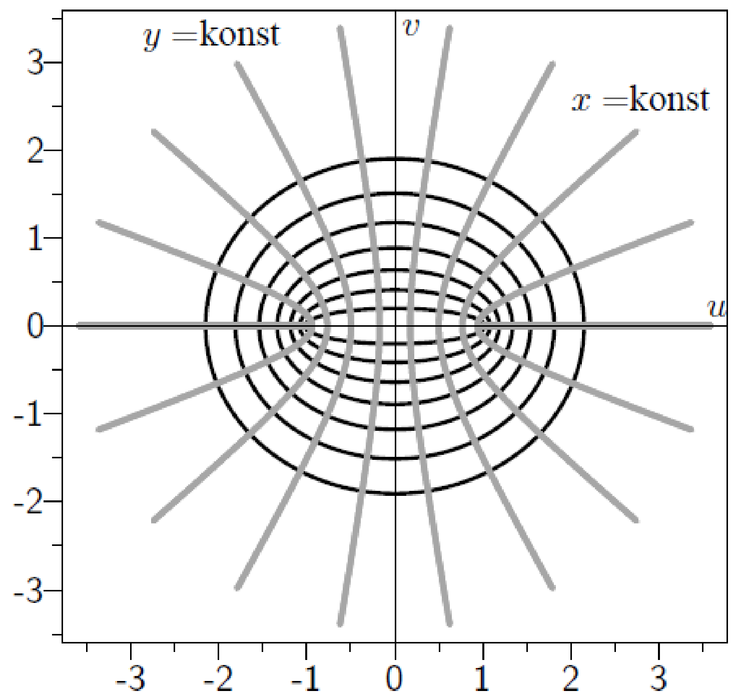
Example: f(z) = cos(z)

$$\cos(z) = \cos(x + \mathrm{i}\, y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

Next thing is the analysis of its geometrical properties:

2. Second, we take a line $x = x_0$, $y \in <-\infty, +\infty>$

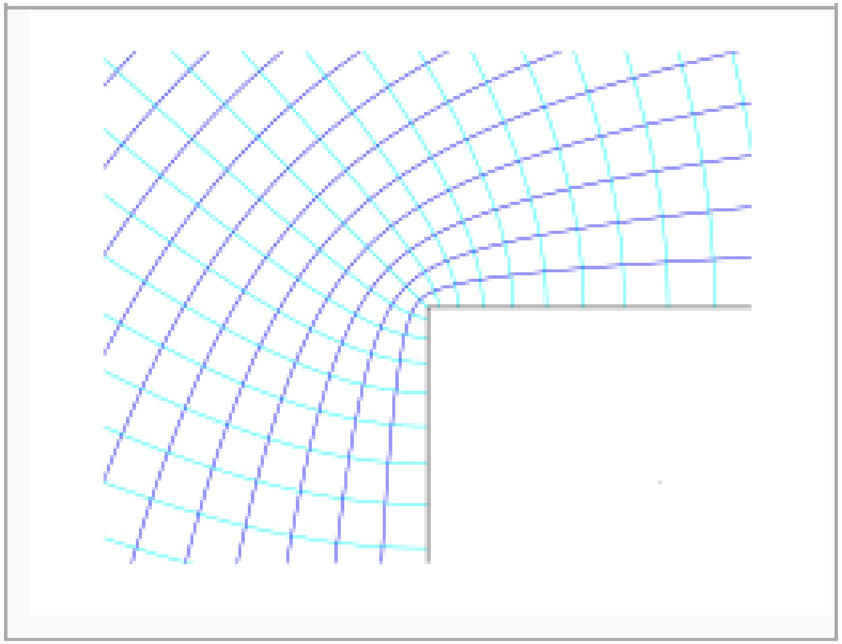




Using $\cosh^2 y - \sinh^2 y = 1$ we get:

$$\frac{u^2}{\cos^2 x_0} - \frac{v^2}{\sin^2 x_0} = 1$$

This is equation of a hyperbole.



model of physical field in close vicinity of an object
(as a result of a solution, involving complex numbers)

Applications, involving complex numbers: improper integrals, geometry, signal analysis, fluid dynamics, dynamic equations, electromagnetism, , quantum mechanics, relativity