Lecture 7: integration – basic concepts and rules

Content:

- integration – introduction, indefinite and definite i.
- integration – calculating integrals
- integration – rules for integration

Integration – introduction

Integration is an opposite operation to differentiation

(summation of infinitesimally small parts of the function $f(x)$) – it is following also from the so called fundamental theorem of calculus (discovered by Newton and Leibniz):

First fundamental theorem of calculus, is that the indefinite integration of a function $f(x)$ is related to its antiderivative.

We recognize two main types of integrals:

a) indefinite integral – the result is a function,

$$\int f(x)\,dx$$

b) definite integral – the results is a number (evaluated indefinite integral at the endpoints of an interval <a, b>: $x = a$ and $x = b$).

$$\int_a^b f(x)\,dx$$

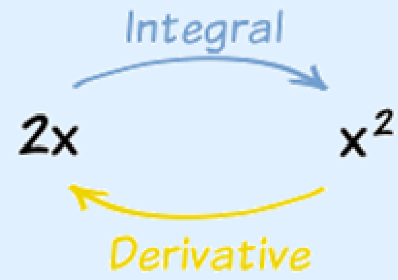
Notation: The integral sign ∫ represents integration. The symbol dx, called the differential of the variable x, indicates that the variable of integration is x. The function $f(x)$ to be integrated is called the integrand. If a function has an integral, it is said to be integrable. The points a and b are the endpoints – called also as the limits (or bounds) of the integral.

Integration – introduction

... finding an Integral is the **reverse** of finding a Derivative.

Example: What is an integral of 2x?

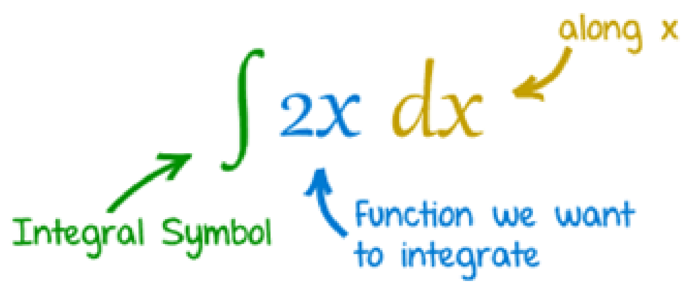
We know that the derivative of x^2 is 2x ...



... so an integral of 2x is x^2

Notation

The symbol for "Integral" is a stylish "S"
(for "Sum", the idea of summing slices):



https://www.mathsisfun.com/calculus/integration-introduction.html

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<u>Comment:</u> The symbol *dx* (differential) is not always placed after the integrand *f*(*x*), as for instance:

$$\int_0^1 \frac{3}{x^2+1}\,dx \quad \text{can be written as:} \quad \int_0^1 \frac{3\,dx}{x^2+1}.$$

Integration – introduction

Roughly speaking, the operation of integration is the reverse of differentiation. The antiderivative, a function *F*(*x*) whose derivative is the integrand *f*(*x*) (also called as primitive function).

1. In the case of an **indefinite integral** we can write:

$$\int f(x)\,dx = F(x)$$

so it is valid: $F'(x) = f(x)$

but at the same time: $(F+c)' = F' = f(x)$ where *c* is a constant, $c \in \mathbf{R}$ (so called arbitrary constant of integration),

so we can write:

$$\int f(x)\,dx = F(x) + c$$

2. In the case of an **definite integral** we can write (on interval <*a*,*b*>):

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

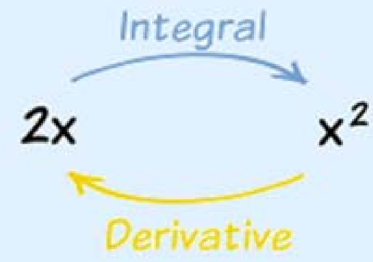
where *F*(*a*) and *F*(*b*) are the primitive functions values for limits *a* and *b*.

Integration – introduction

Back to our simple example:

Example: What is an integral of 2x?

We know that the derivative of x^2 is 2x ...



... so an integral of 2x is x^2

Notation

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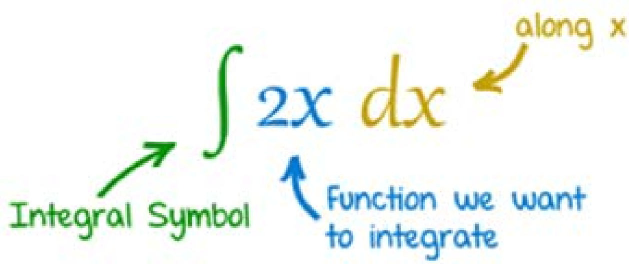
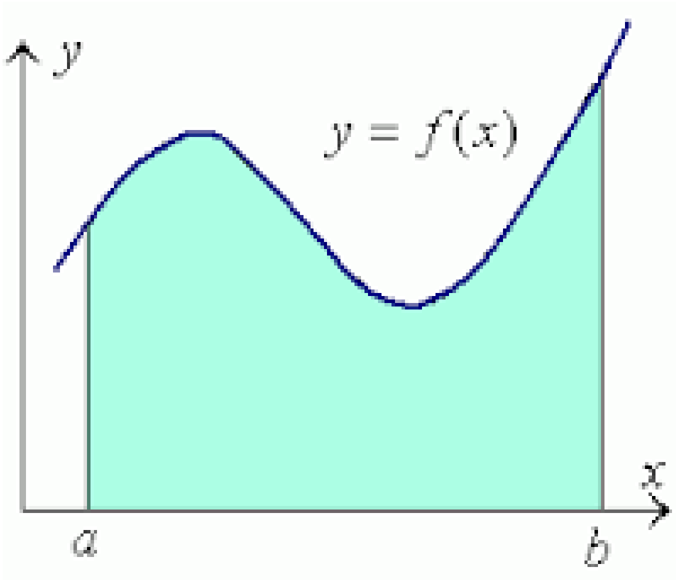


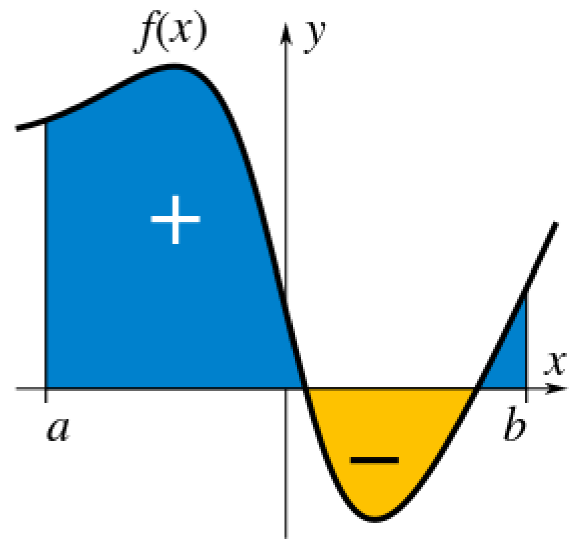
... so we should write: $\int 2x\,dx = x^2 + C$

$$\frac{d}{dx}(x^2 + C) = 2x$$

Integration – introduction

Geometrical meaning of integration:





Definite integral of function *f*(*x*) on the interval <*a*,*b*> corresponds to the size of the area between the graph of function *f*(*x*) and the horizontal x-axis (limited by limits *a* and *b*).

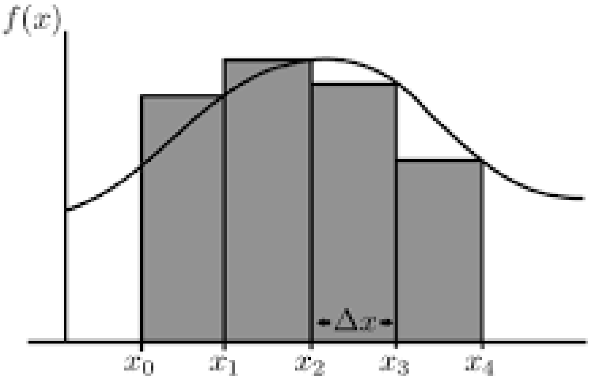
Important: The area can be also negative – it is dependent on the signs of functional values of the intergrated function *f*(*x*).

This aspect of integration has many kinds of application in applied mathematics, physics and technics (and not only in 1 D!).

Integration – introduction

A simple "definition" of the definite integral is based on the limit of a Riemann sum of right rectangles. The exact area under a curve between *a* and *b* is given by the definite integral, which is defined as follows:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i)\cdot\left(\frac{b-a}{n}\right)\right]$$



$$dx \approx \Delta x = \frac{b-a}{n}$$

To make the approximating region you choose a *partition* of the interval $[a, b]$, i.e. you pick numbers $x_1 < \cdots < x_n$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These numbers split the interval $[a, b]$ into n sub-intervals

$$[x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{n-1}, x_n]$$

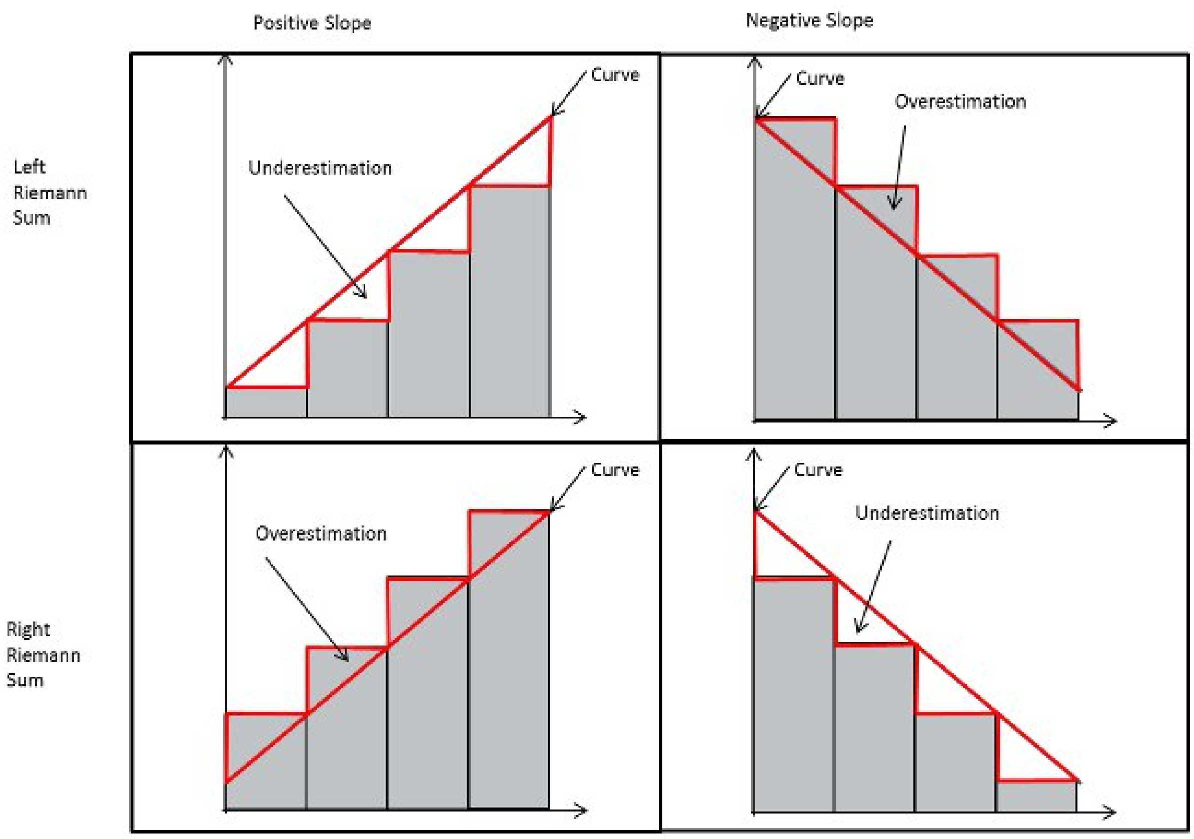
Integration – introduction

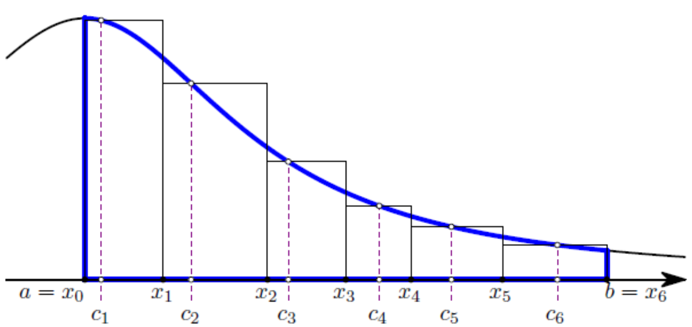
Limit of a Riemann sum of right rectangles:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i)\cdot\left(\frac{b-a}{n}\right)\right]$$

There can originate 3 different situations:

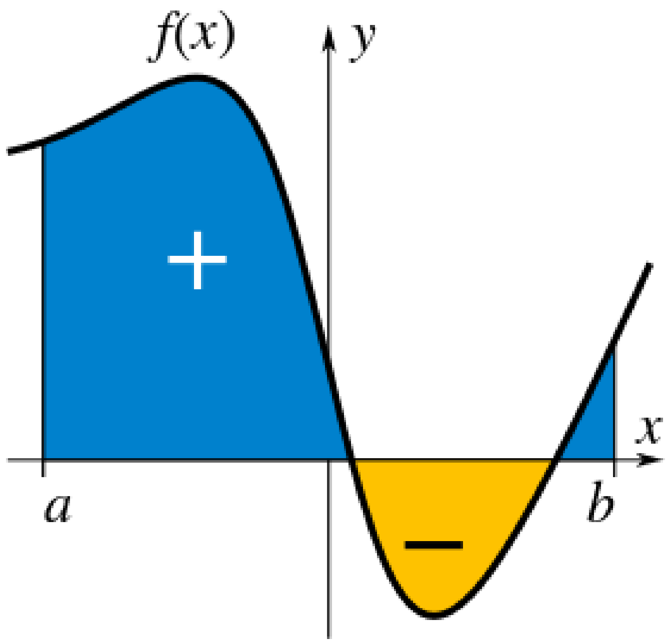
- so called left Riemann sum,
- so called right Riemann sum,
- and so called midpoint Riemann sum.

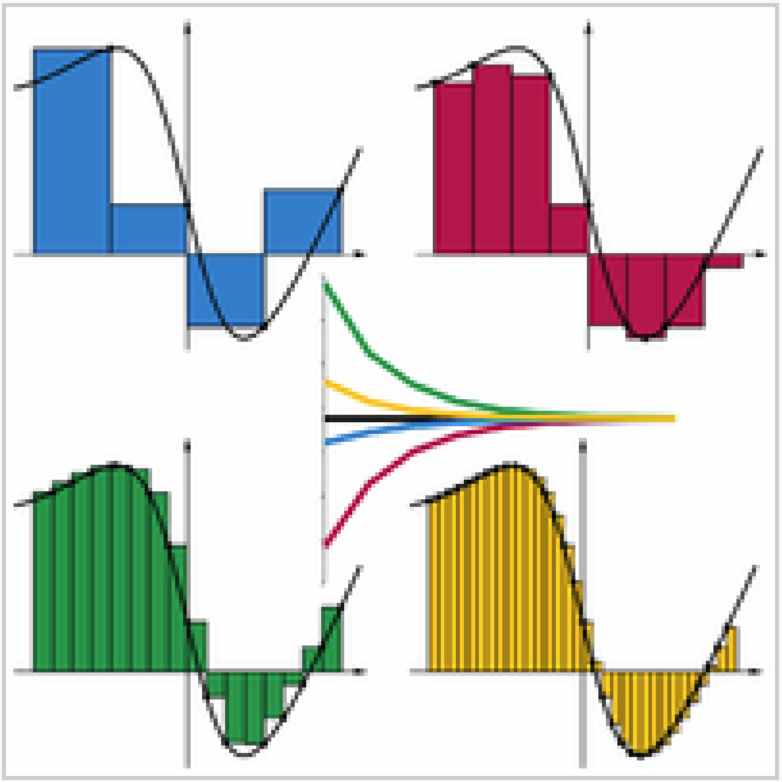




midpoint Riemann sum

Integration – introduction





Riemann sums converging

There exists very similar definition of an integral – based on so called Darboux sums (different way of rectangles “organisation”).

(often named as Riemann-Darboux approach).

Nice visualization:

https://upload.wikimedia.org/wikipedia/commons/thumb/0/0a/Riemann_Integration_and_Darboux_Upper_Sums.gif/300px-Riemann_Integration_and_Darboux_Upper_Sums.gif

Integration – introduction

Definition of a definite integral in the sense of Riemann:

Definition. *If f is a function defined on an interval $[a,b]$, then we say that*

$$\int_a^b f(x)dx = I,$$

i.e. the integral of "$f(x)$ from $x = a$ to b" equals I, if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\left| f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n - I \right| < \varepsilon$$

holds for every partition all of whose intervals have length $\Delta x_k < \delta$.

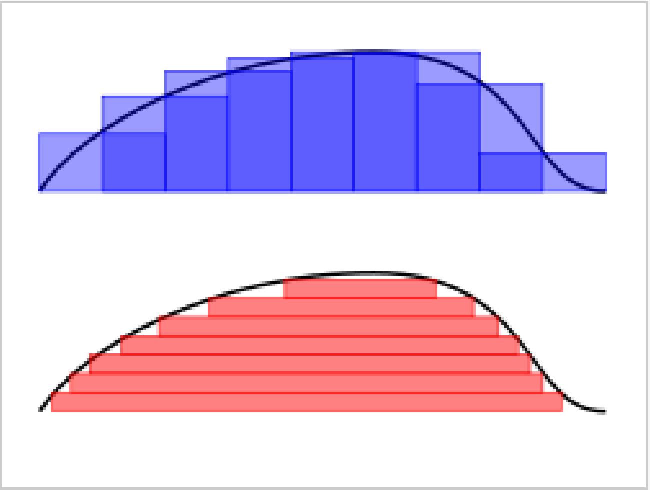
Comment: There are many functions that can be obtained as limits – these are not Riemann-integrable.

Integration – introduction

From this need, there are also other definitions of integrals – the well known is the integral in the sense of Lebesgue:

Citation from text-book of Folland (1984):

"To compute the Riemann integral of function *f*, one partitions the domain [a, b] into subintervals", while in the Lebesgue integral, "one is in effect partitioning the range of function *f*".



Riemann–Darboux's integration (top) and Lebesgue integration (bottom)

Henri Lebesgue introduced his kind of integral in a letter to the mathematician Paul Montel:

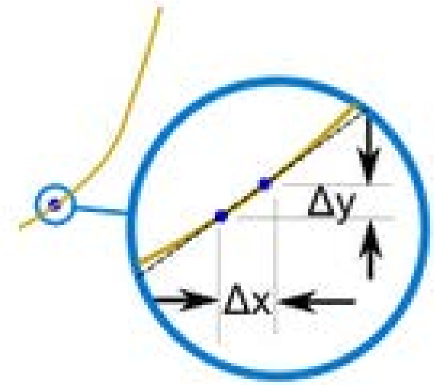
“I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.”

Integration – introduction

Final comment from the introduction:

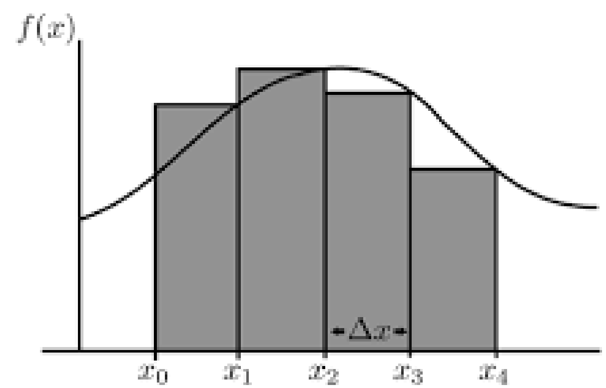
differentiation vs. integration (often called as “calculus”)

$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



differentiation – very small changes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i) \cdot \left(\frac{b-a}{n} \right) \right]$$



integration – summation of very small areas (contributions)

Lecture 7: integration – basic concepts and rules

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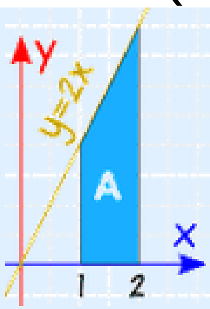
Integration – calculating integrals

Again (like in the case of limits and derivatives evaluation), the direct use of definition for definite integral is too cumbersome, to be used in usual calculation of integrals.

But one example, where also the definition is used (1/2):

Example:

The **Definite Integral**, from 1 to 2, of **2x** dx:



$$\int_1^2 2x\,dx$$

The ***Indefinite* Integral** is: $\int 2x\,dx = x^2 + C$

- At x=1: $\int 2x\,dx = 1^2 + C$
- At x=2: $\int 2x\,dx = 2^2 + C$

Subtract:

→ $(2^2 + C) - (1^2 + C)$

→ $2^2 + C - 1^2 - C$

→ $4 - 1 + C - C = 3$

And "C" gets cancelled out ... so with Definite Integrals we can ignore C.

Comment: here we use the antiderivative (primitive function).

Integration – calculating integrals

Again (like in the case of limits and derivatives evaluation), the direct use of definition for definite integral is too cumbersome, to be used in usual calculation of integrals.

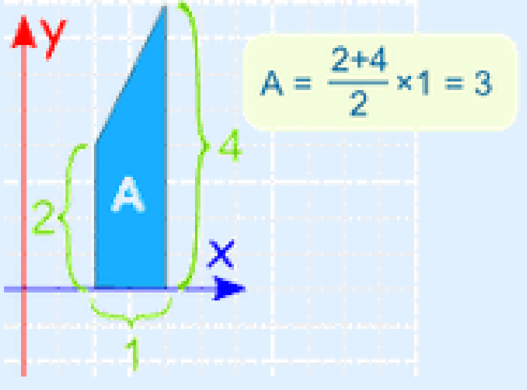
But one example, where also the definition is used (2/2):

In fact we can give the answer directly like this:

$$\int_1^2 2x\,dx = 2^2 - 1^2 = 3$$

We can check that, by calculating the area of the shape:

Yes, it has an area of 3.



Comment: here we have used the formula for the trapezoid area evaluation.

Integration – calculating integrals

The commonly used method for definite integrals evaluation is using the antiderivative (primitive function):

Definition. *A function F is called an antiderivative of f on the interval $[a, b]$ if one has $F'(x) = f(x)$ for all x with $a < x < b$.*

For instance, $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$, but so is $G(x) = \frac{1}{2}x^2 + 2008$.

Theorem. *If f is a function whose integral $\int_a^b f(x)dx$ exists, and if F is an antiderivative of f on the interval $[a, b]$, then one has*

$$\int_a^b f(x)dx = F(b) - F(a).$$

It is interesting that this rule is working also in a situation when *f*(*a*) and/or *f*(*b*) is negative value.

We can see that for the evaluation of definite integrals we need to find primitive functions of indefinite integrals – so firstly, we will focus in the lecture on the methods of indefinite integrals evaluation.

Integration – calculating integrals

Another viewpoint to the two most important kind of integrals (indefinite and definite integrals):

INDEFINITE INTEGRAL	DEFINITE INTEGRAL
$\int f(x)dx$ is a function of x.	$\int_a^b f(x)dx$ is a number.
By definition $\int f(x)dx$ is *any function of x whose derivative is $f(x)$*.	$\int_a^b f(x)dx$ was defined in terms of Riemann sums and can be interpreted as "area under the graph of $y = f(x)$", at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2xdx = x^2 + C$ and $\int 2tdt = t^2 + C$ are functions of different variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2xdx = 1$, and $\int_0^1 2tdt = 1$, so $\int_0^1 2xdx = \int_0^1 2tdt$.

Integration – calculating integrals

Basic „rule" for the solution of indefinite integrals is:

„good knowledge of differentiation!!!"

(integration of basic functions is simply an opposite operation to the differentiation).

But there exist few methods, which can help...

It is good always to check the result of integration – by means of differentiation (as a test).

Important! Not every integral can be solved – in contrary to the differentiation, where almost all functions can be differentiated.

$$\frac{dc}{dx} = 0 \qquad \frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}\cos x = -\sin x$$

Integration – calculating integrals

From the reason that integration is a much more complicated operation than differentiation, there exist special tables (some of them are quite rich – on of the most known are the tables: Gradshtein, I. S., Ryzhik, I. M. 1994,Tables of integrals, series and products, 5th ed.Academic Press)

Common Functions	Function	Integral
Constant	$\int a\,dx$	$ax + C$
Variable	$\int x\,dx$	$x^2/2 + C$
Square	$\int x^2\,dx$	$x^3/3 + C$
Reciprocal	$\int (1/x)\,dx$	$\ln\lvert x\rvert + C$
Exponential	$\int e^x\,dx$	$e^x + C$
	$\int a^x\,dx$	$a^x/\ln(a) + C$
	$\int \ln(x)\,dx$	$x\ln(x) - x + C$
Trigonometry (x in radians)	$\int \cos(x)\,dx$	$\sin(x) + C$
	$\int \sin(x)\,dx$	$-\cos(x) + C$
	$\int \sec^2(x)\,dx$	$\tan(x) + C$
Rules	**Function**	**Integral**
Multiplication by constant	$\int cf(x)\,dx$	$c\int f(x)\,dx$
Power Rule ($n\neq-1$)	$\int x^n\,dx$	$x^{n+1}/(n+1) + C$
Sum Rule	$\int (f + g)\,dx$	$\int f\,dx + \int g\,dx$
Difference Rule	$\int (f - g)\,dx$	$\int f\,dx - \int g\,dx$
Integration by Parts	See Integration by Parts	
Substitution Rule	See Integration by Substitution	

Integration – calculating integrals

It is always important to write in the result of indefinite integration the arbitrary integration constant C, because without it confusing situations can occur, e.g.:

$$\int 2\sin x\cos x\,dx = \sin^2 x$$

$$\int 2\sin x\cos x\,dx = -\cos^2 x$$

because: $\sin^2 x = -\cos^2 x + 1$

from it follows that in the first integral $C = -1$ and in the second $C = 1$.

To avoid this kind of confusion we will from now on never forget to include the "arbitrary constant $+C$" in our answer when we compute an antiderivative.

Lecture 7:   integration – basic concepts and rules

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Integration – calculating integrals

Rules for integration – in details:

1. Multiplication by a constant
2. Power rule
3. Sum rule
4. Difference rule
5. Integration by parts
   (so called "per partes" method)
6. Substitution rule

Function	Integral
$\int cf(x)\,dx$	$c\int f(x)\,dx$
$\int x^n\,dx$	$x^{n+1}/(n+1) + C$
$\int (f + g)\,dx$	$\int f\,dx + \int g\,dx$
$\int (f - g)\,dx$	$\int f\,dx - \int g\,dx$
See Integration by Parts	
See Integration by Substitution	

Rules for integration – in details:

1. Multiplication by a constant:

It is coming from the derivative of a function multiplied by a constant:

$$\frac{d}{dx}\left(cF(x)\right) = cF'(x) = cf(x) \text{ implies that}$$

↓

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Example:

$$\int \frac{3}{x}dx = 3\int \frac{1}{x}dx = 3\ln x + C$$

Rules for integration – in details:

2. Power rule:

It is coming from the power rule for derivative:

$$\frac{d}{dx} x^{n} = n x^{n-1}$$

$$\downarrow$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for all } n \neq -1$$

But what is the solution for $n = -1$?

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Example:

$$\int x^4 dx = \frac{x^5}{5} + C$$

Rules for integration – in details:

3. and 4. Sum and difference rule:

It is coming from the sum and difference rule for derivatives evaluation:

Sum rule: $(u \pm v)' = u' \pm v'$

↓

$$\int \{f(x) \pm g(x)\}\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Example:

$$\int (\cos x + \sin x) dx = \int \cos x dx + \int \sin x dx = \sin x - \cos x + C$$

Rules for integration – in details:

5. Integration by parts (so called "per partes" method):

This theorem relates the integral of a product of functions to the integral of their derivative and original function.

If $u = u(x)$ and $du = u'(x)\,dx$, while $v = v(x)$ and $dv = v'(x)\,dx$, then integration by parts states that:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)\,u'(x)dx$$

or more compactly:

$$\int u\,v'\,dx = u\,v - \int u'v\;dx$$

PROOF is derived from the product rule for derivatives:

$$\textit{Product rule:} \qquad (u \cdot v)' = u' \cdot v + u \cdot v'$$

After integration both sides:

$$\int (u \cdot v)'\,dx = \int u' \cdot v\,dx + \int u \cdot v'\,dx$$

and rearranging of the terms yields:

$$\int u \cdot v'\,dx = (u \cdot v) - \int u' \cdot v\,dx$$

Rules for integration – in details:

5. Integration by parts (so called “per partes” method):

Example:

We have to solve the following integral (a product of two functions):

$$\int x\cos(x)dx = ?$$

The formulation of the per partes integration is

(setting exactly the functions *u* and *v* and their derivatives *u*’ and *v*’):

$$\int x\cos(x)dx \begin{bmatrix} u = x & v' = \cos(x) \\ u' = 1 & v = \sin(x) \end{bmatrix} = x\sin(x) - \int \sin(x)dx =$$

$$\boxed{\int x\cos(x)dx = x\sin(x) + \cos(x) + C}$$

where C is the constant of integration.

Rules for integration – in details:

5. Integration by parts (so called “per partes” method):

Example:

We have to solve the next integral – from ArcTan(x) = $\tan^{-1}(x)$:

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln \left|1+x^2\right| + C$$

$$= x \tan^{-1} x - \frac{1}{2} \ln \left(1+x^2\right) + C$$

Comment: In the last step we have written instead of the absolute value in the natural logarithm function its value in parentheses, because it is always positive.

Rules for integration – in details:

5. Integration by parts (so called "per partes" method):

Example: We have to solve the following integral:

$$\int x^2 \cos(x)\,dx =$$

$$u = x^2,\; du = 2x\,dx$$

$$dv = \cos(x)\,dx\;,\; v = \sin(x)$$

$$\int x^2 \cos(x)\,dx = x^2 \sin(x) - \left[\int 2x \sin(x)\,dx\right]$$

$$\int 2x \sin(x)\,dx = 2x(-\cos(x)) - \int 2(-\cos(x))\,dx$$

$$\int 2(-\cos(x))\,dx = -2\sin(x)$$

$$\int x^2 \cos(x)\,dx = x^2 \sin(x) - 2x(-\cos(x)) - (2\sin(x))$$

$$= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + c$$

Comment: We can apply the integration by parts several times in a sequence.

Rules for integration – in details:

6. Substitution rule:

Integration by substitution (called also as u-substitution or simply the substitution method) — is a technique of integration whereby a complicated looking integrand is rewritten into a simpler form by using a change of variables:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \text{ where } u = g(x).$$

It follows also from the chain rule:

The chain rule says that

$$\frac{dF(G(x))}{dx} = F'(G(x)) \cdot G'(x),$$

so that

$$\int F'(G(x)) \cdot G'(x)\, dx = F(G(x)) + C.$$

Rules for integration – in details:

6. Substitution rule: $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, where $u = g(x)$.

Example:

Example

Consider the integral:

$$\int x(x+3)^7\,dx$$

By letting $u = x + 3$, thus $du = dx$ (since $\frac{du}{dx} = 1$), and observing that $x = u - 3$, the integral simplifies to

$$\int x(x+3)^7\,dx = \int (u-3)u^7\,du = \int (u^8 - 3u^7)\,du,$$

which is easily integrated to obtain:

$$\frac{u^9}{9} - \frac{3u^8}{8} + C = \frac{(x+3)^9}{9} - \frac{3(x+3)^8}{8} + C.$$

Note that this integral can also be done using integration by parts, although the final answer may look different because of the different steps involved.

Rules for integration – in details:

6. Substitution rule: $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, where $u = g(x)$.

Example:

Solve the following integral:

$$\int \frac{b}{(x-a)^2+b^2}\,dx = b\int \frac{dx}{(x-a)^2+b^2} =$$

$$= b\int \frac{\frac{dx}{b^2}}{\frac{(x-a)^2}{b^2}+\frac{b^2}{b^2}} = \frac{b}{b^2}\int \frac{dx}{\left(\frac{x-a}{b}\right)^2+1}\begin{bmatrix} t=(x-a)/b \\ dt = dx/b \\ dx = b\,dt \end{bmatrix} =$$

$$= \frac{1}{b}\int \frac{b\,dt}{t^2+1} = \int \frac{1}{t^2+1}\,dt = artctg(t)+C = artctg\left(\frac{x-a}{b}\right)+C$$

Rules for integration – in details:

6. Substitution rule: $$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \text{ where } u = g(x).$$

Example:

Example

To compute $\int \sin(2x)\cos(2x)\,dx$, let

$$u = \sin(2x)$$
$$du = 2\cos(2x)\,dx.$$

Then

$$\int \sin(2x)\cos(2x)\,dx = \int \frac{1}{2}\sin(2x)[2\cos(2x)\,dx] = \int \frac{1}{2}u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\sin^2(2x) + C.$$

Rules for integration:

So called nonelementary integrals.

Can not be solved in real numbers domain with the help

of these simple rules. Some of them can be solved with the help of Taylor series or with complex number functions.

Some examples of such functions are:

- $\sqrt{1-x^4}$ [1]
- $\ln(\ln x)$
- $\frac{1}{\ln x}$ [3] (see Logarithmic integral)
- $\frac{e^x}{x}$ (see Exponential integral)
- e^{e^x}
- $e^{-\frac{x^2}{2}}$ [1] (see Normal distribution)
- $\sin(x^2)$ and $\cos(x^2)$ (see Fresnel integral)

Some indefinite integrals have no solutions
(utilising in the primitive function known elementary functions).

Example:

substituion method

$$\int \sin\left(x^2\right)dx \left[\begin{matrix} t = x^2, x = \sqrt{t} \\ dt = 2xdx, dx = dt/\left(2\sqrt{t}\right) \end{matrix}\right] =$$

per-partes method

$$= \int \frac{\sin t}{2\sqrt{t}}dt \left[\begin{matrix} u' = 1/\left(2\sqrt{t}\right), v = \sin t \\ u = \sqrt{t}, \qquad v' = \cos t \end{matrix}\right] = \sqrt{t}\sin t - \int \sqrt{t}\cos x dx =$$

... used again

$$= \sqrt{t}\sin t - \int \sqrt{t}\cos t dt \left[\begin{matrix} u = \sqrt{t}, \qquad v' = \cos t \\ u' = 1/\left(2\sqrt{t}\right), v = \sin t \end{matrix}\right] =$$

$$= \sqrt{t}\sin t - \left[\sqrt{t}\sin t - \int \frac{\sin t}{2\sqrt{t}}dt\right] = \int \frac{\sin t}{2\sqrt{t}}dt = \ldots$$

We call such a function as <u>a function with non-elementary integral</u>.