Lecture 8: definite integrals, properties, applications

Content:

- indefinite and definite integrals, examples
  (little bit repetition from previous lecture plus new facts)
- some properties of definite integrals
- improper integrals
- applications of definite integrals

Integration – repetition (previous lecture)

Integration is an opposite operation to differentiation

(summation of infinitesimally small parts of the function *f*(*x*)) –

First fundamental theorem of calculus: the indefinite integration of a function *f*(*x*) is related to its antiderivative.

We recognize two main types of integrals:

a) indefinite integral – the result is a function,

$$\int f(x)\,dx$$

b) definite integral – the results is a number (evaluated indefinite integral at the endpoints of an interval <*a*, *b*>: *x* = *a* and *x* = *b*).

$$\int_a^b f(x)\,dx$$

Notation: The integral sign ∫ represents integration. The symbol *dx*, called the differential of the variable *x*, indicates that the variable of integration is *x*. The function *f*(*x*) to be integrated is called the integrand. If a function has an integral, it is said to be integrable. The points *a* and *b* are the endpoints – called also as the limits (or bounds) of the integral.

Integration – repetition (previous lecture)

Roughly speaking, the operation of integration is the reverse of differentiation. The antiderivative, a function *F*(*x*) whose derivative is the integrand *f*(*x*) (also called as primitive function).

1. In the case of an **indefinite integral** we can write:

$$\int f(x)\,dx = F(x)$$

so it is valid: $F'(x) = f(x)$

but at the same time: $(F+c)' = F' = f(x)$ where *c* is a constant, $c \in \mathbf{R}$ (so called arbitrary constant of integration),
so we can write:

$$\int f(x)\,dx = F(x) + c$$

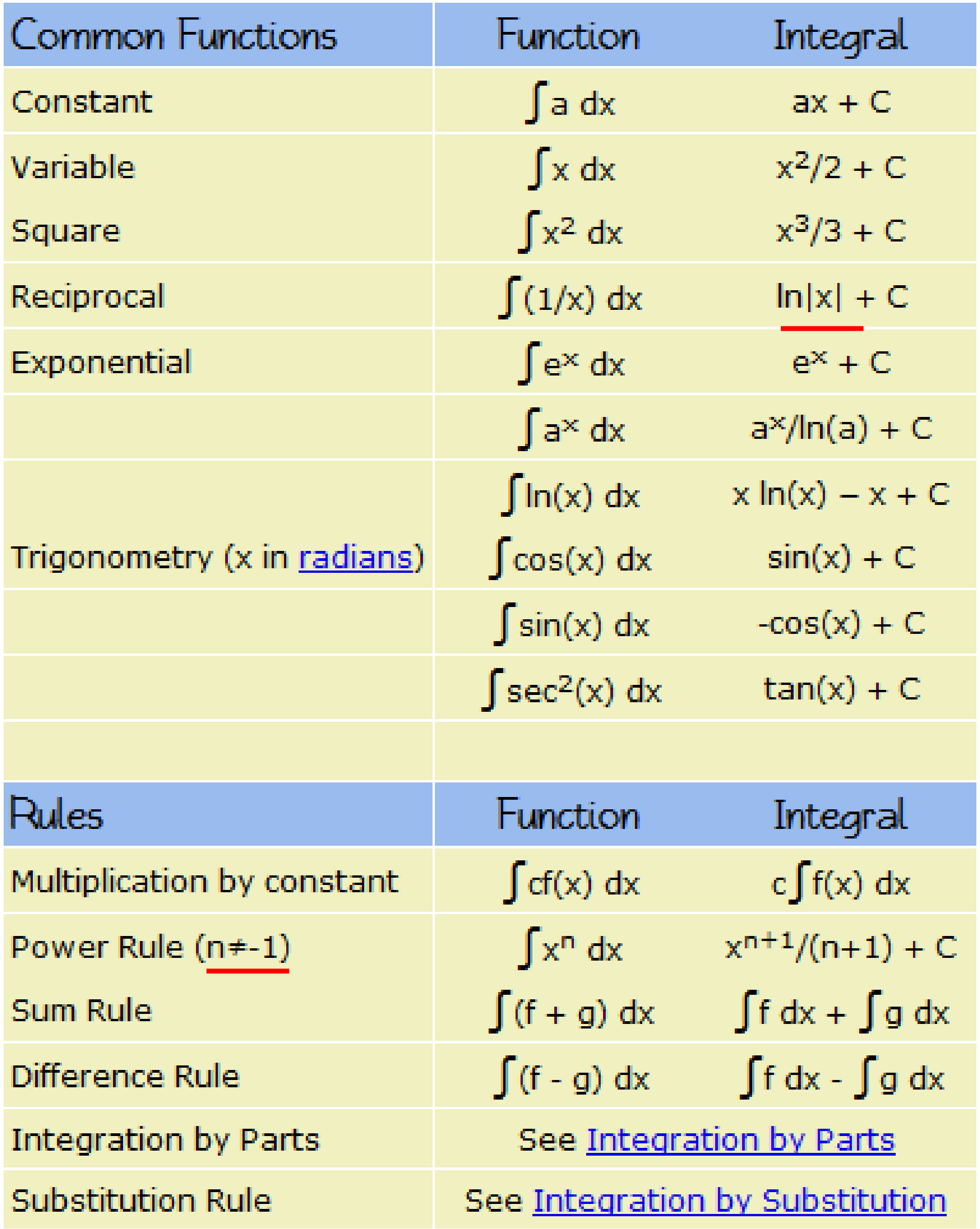
2. In the case of an **definite integral** we can write (on interval <*a*,*b*>):

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where *F*(*a*) and *F*(*b*) are the primitive functions values for limits *a* and *b*.

to calculate definite integrals we have first to evaluate indefinite integrals...

repetition from previous lecture:



Common Functions	Function	Integral
Constant	$\int a\,dx$	$ax + C$
Variable	$\int x\,dx$	$x^2/2 + C$
Square	$\int x^2\,dx$	$x^3/3 + C$
Reciprocal	$\int (1/x)\,dx$	$\ln\lvert x\rvert + C$
Exponential	$\int e^x\,dx$	$e^x + C$
	$\int a^x\,dx$	$a^x/\ln(a) + C$
	$\int \ln(x)\,dx$	$x\ln(x) - x + C$
Trigonometry (x in radians)	$\int \cos(x)\,dx$	$\sin(x) + C$
	$\int \sin(x)\,dx$	$-\cos(x) + C$
	$\int \sec^2(x)\,dx$	$\tan(x) + C$
Rules	**Function**	**Integral**
Multiplication by constant	$\int cf(x)\,dx$	$c\int f(x)\,dx$
Power Rule ($n\neq-1$)	$\int x^n\,dx$	$x^{n+1}/(n+1) + C$
Sum Rule	$\int (f + g)\,dx$	$\int f\,dx + \int g\,dx$
Difference Rule	$\int (f - g)\,dx$	$\int f\,dx - \int g\,dx$
Integration by Parts	See Integration by Parts	
Substitution Rule	See Integration by Substitution	

Lecture 8: definite integrals, properties, applications

- definite integrals, examples

Definite integrals – simple examples:

Example

$$\int_2^3 x^3\,dx = \tfrac{1}{4}\,x^4\Big|_2^3 = \tfrac{1}{4}\,3^4 - \tfrac{1}{4}\,2^4 = \tfrac{81}{4} - \tfrac{16}{4} = \tfrac{65}{4} = 16.25$$

This notation means: evaluate the function at 3 and 2, and subtract the results.

Example

$$\int_0^{12} 3x + 5\,dx = 276$$

An antiderivative for $f(x) = 3x + 5$ is $F(x) = \tfrac{3}{2}x^2 + 5x$

So: $$\int_0^{12} 3x + 5\,dx = \mathrm{F}(12) - \mathrm{F}(0) = 276 - 0 = 276$$

Definite integrals – simple examples:

Example:

$$\int_0^{\pi} \sin(x)dx = [-\cos(x)]_0^{\pi} = -[-1-1] = 2$$

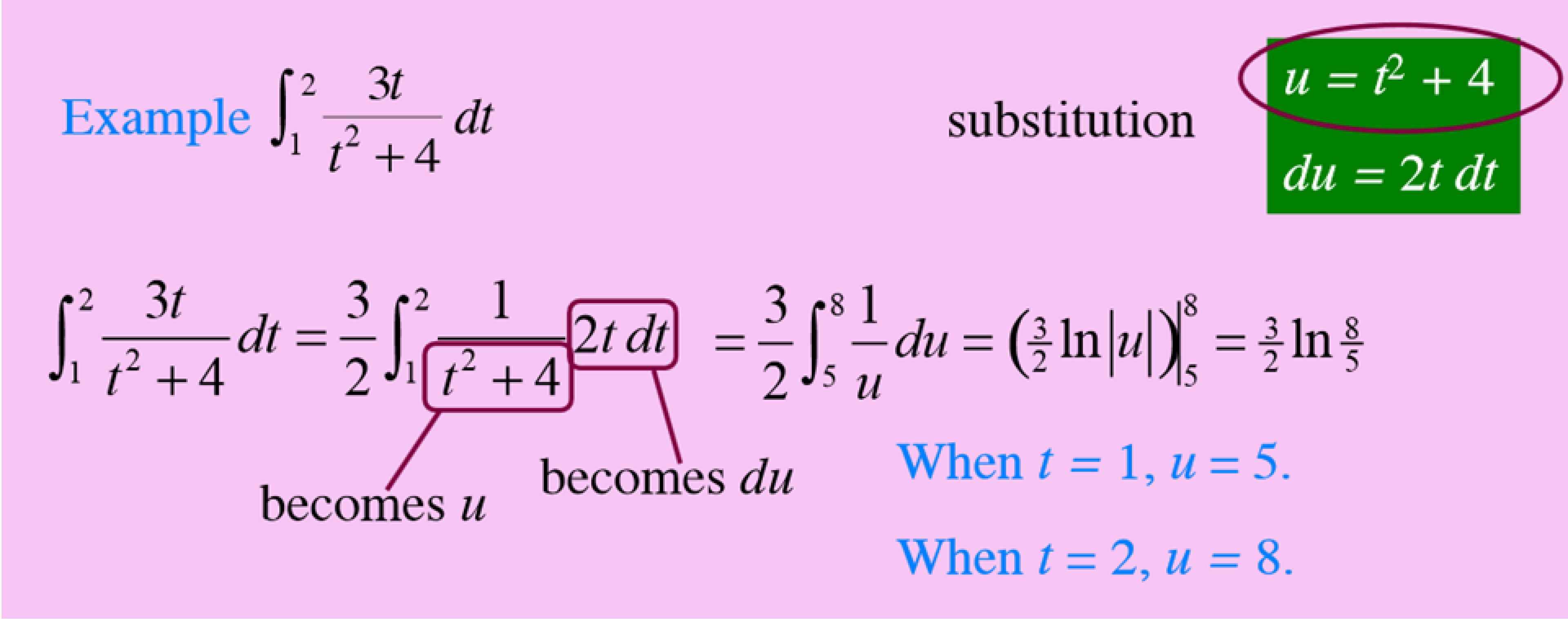
Example:

$$\int_{-2}^{-1} \frac{xe^x+1}{x}dx = \int_{-2}^{-1} e^x dx + \int_{-2}^{-1} \frac{1}{x}dx = \left(e^x + ln|x|\right)\Big|_{-2}^{-1} = \left(e^{-1}+ln1\right) - \left(e^{-2}+ln2\right) =$$

$$= \left(\frac{1}{e}+0\right) - \left(\frac{1}{e^2}+ln2\right) = \frac{1}{e} - \frac{1}{e^2} - ln2$$

But what to do in the case of the use of substitution method?

During the application of substitution we change the integration variable – so we should also change the actual limits or solve this problem in the end after the back-substitution.



Example $\int_1^2 \frac{3t}{t^2+4}\,dt$

substitution

$u = t^2 + 4$

$du = 2t\,dt$

$$\int_1^2 \frac{3t}{t^2+4}\,dt = \frac{3}{2}\int_1^2 \frac{1}{\boxed{t^2+4}}\,\boxed{2t\,dt} = \frac{3}{2}\int_5^8 \frac{1}{u}\,du = \left(\tfrac{3}{2}\ln|u|\right)\Big|_5^8 = \tfrac{3}{2}\ln\tfrac{8}{5}$$

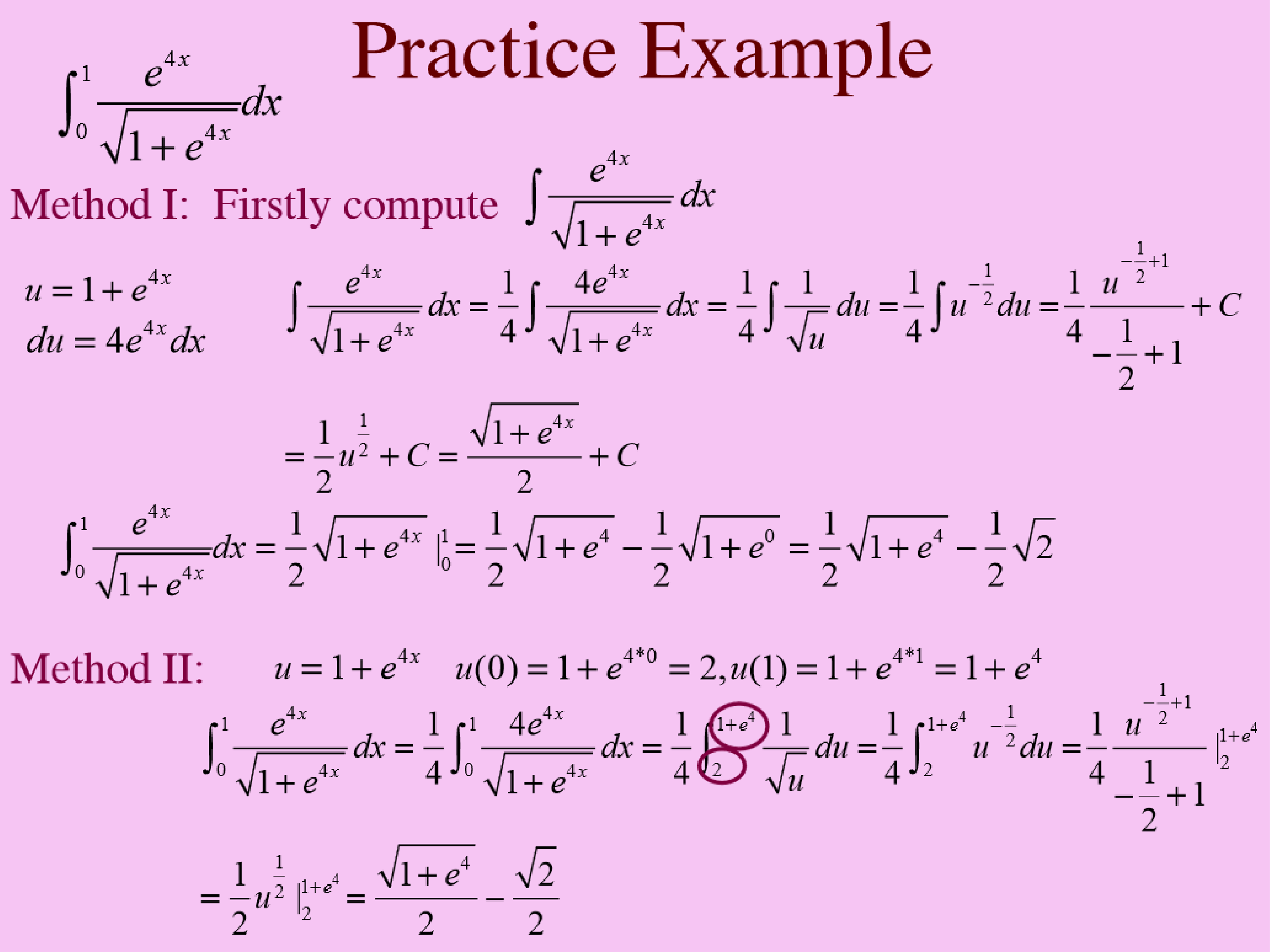
becomes u

becomes du

When $t = 1$, $u = 5$.

When $t = 2$, $u = 8$.

But what to do in the case of the use of substitution method?



Practice Example

$$\int_0^1 \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx$$

Method I: Firstly compute $\int \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx$

$u = 1+e^{4x}$

$du = 4e^{4x}dx$

$$\int \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx = \frac{1}{4}\int \frac{4e^{4x}}{\sqrt{1+e^{4x}}}dx = \frac{1}{4}\int \frac{1}{\sqrt{u}}du = \frac{1}{4}\int u^{-\frac{1}{2}}du = \frac{1}{4}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$$

$$= \frac{1}{2}u^{\frac{1}{2}}+C = \frac{\sqrt{1+e^{4x}}}{2}+C$$

$$\int_0^1 \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx = \frac{1}{2}\sqrt{1+e^{4x}}\,\Big|_0^1 = \frac{1}{2}\sqrt{1+e^4} - \frac{1}{2}\sqrt{1+e^0} = \frac{1}{2}\sqrt{1+e^4} - \frac{1}{2}\sqrt{2}$$

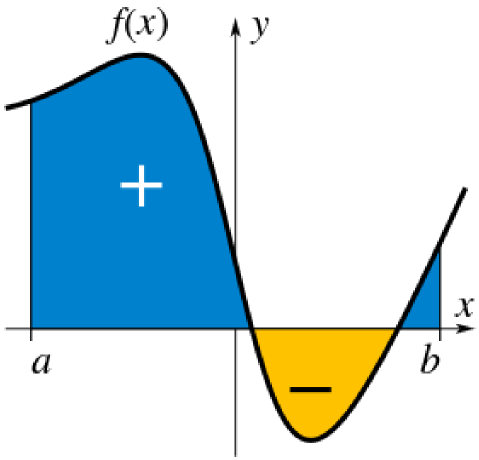
Method II: $u = 1+e^{4x} \quad u(0) = 1+e^{4*0} = 2, u(1) = 1+e^{4*1} = 1+e^4$

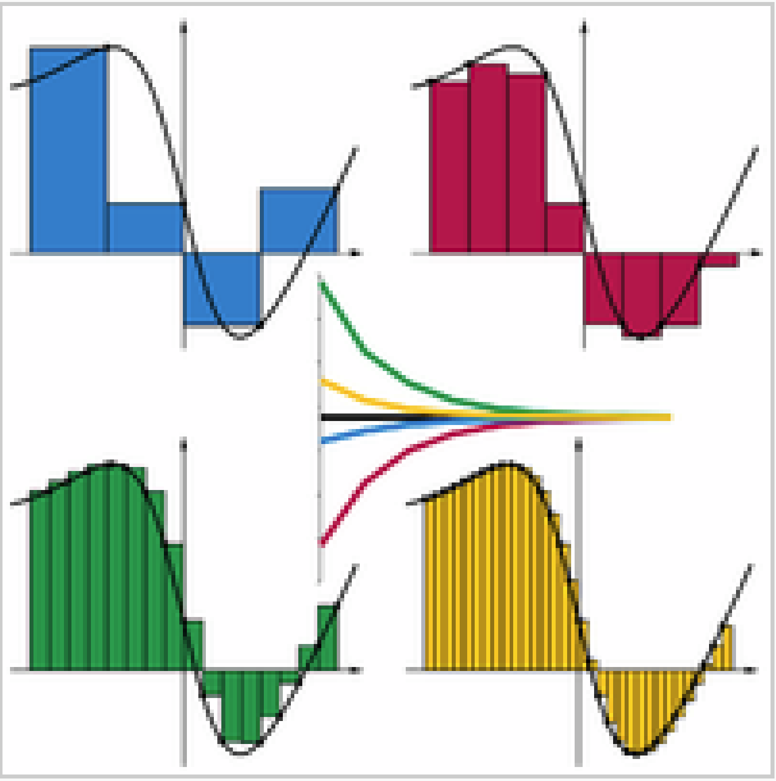
$$\int_0^1 \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx = \frac{1}{4}\int_0^1 \frac{4e^{4x}}{\sqrt{1+e^{4x}}}dx = \frac{1}{4}\int_{\boxed{2}}^{\boxed{1+e^4}} \frac{1}{\sqrt{u}}du = \frac{1}{4}\int_2^{1+e^4} u^{-\frac{1}{2}}du = \frac{1}{4}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Big|_2^{1+e^4}$$

$$= \frac{1}{2}u^{\frac{1}{2}}\Big|_2^{1+e^4} = \frac{\sqrt{1+e^4}}{2} - \frac{\sqrt{2}}{2}$$

Comment to the geometrical meaning:

Definite integral is equal to the area under a curve between bounds *a* and *b* – so called a Riemann sum of right rectangles:

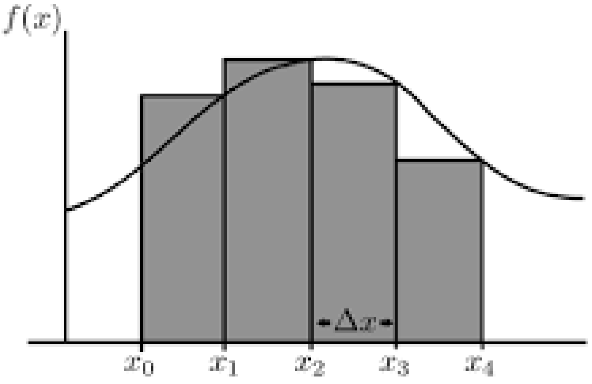




Riemann sums converging

formally written as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i)\cdot\left(\frac{b-a}{n}\right)\right]$$

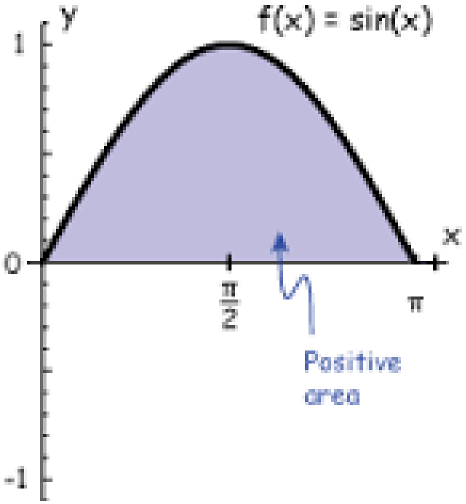


Comment to the geometrical meaning:

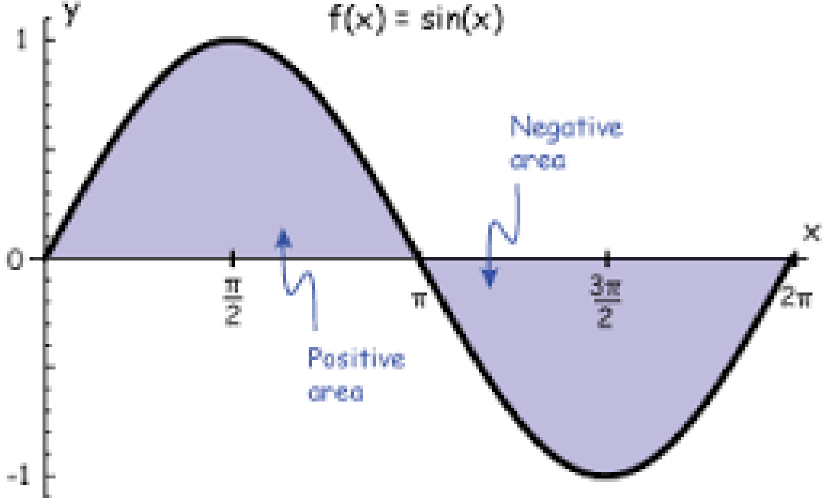
Definite integral is equal to the area under a curve between bounds *a* and *b* – so called a Riemann sum of right rectangles:

Examples:

$$\int_0^{\pi} \sin(x)dx = [-\cos(x)]_0^{\pi} = -[-1-1] = 2$$



$$\int_0^{2\pi} sin(x)dx = [-cos(x)]_0^{2\pi} = -[1-1] = 0$$



Comment to the geometrical meaning:

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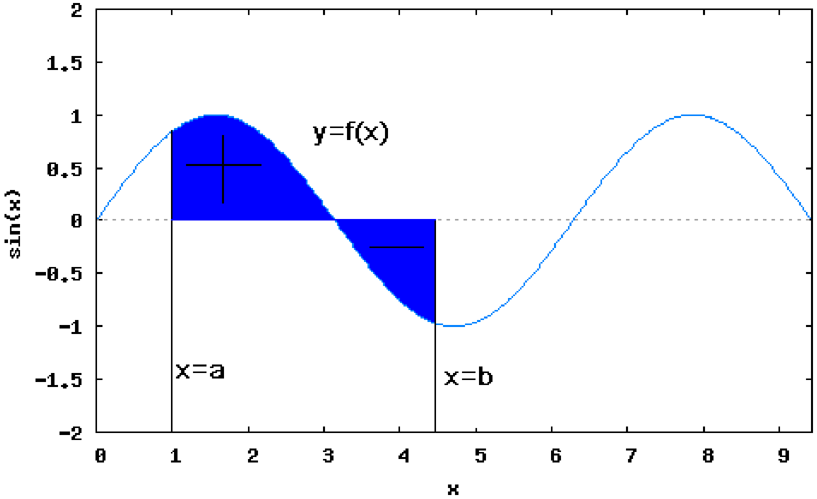

Comment: Entering the limits into the primitive function solves the problem with the correct signs of the involved (appropriate) areas.

Exercise with MATLAB: To show the integration of sin-function, using the *trapz*() built-in function of Matlab (numerical method for integration).

radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
cos x	1	0.866	0.707	0.500	0	−0.500

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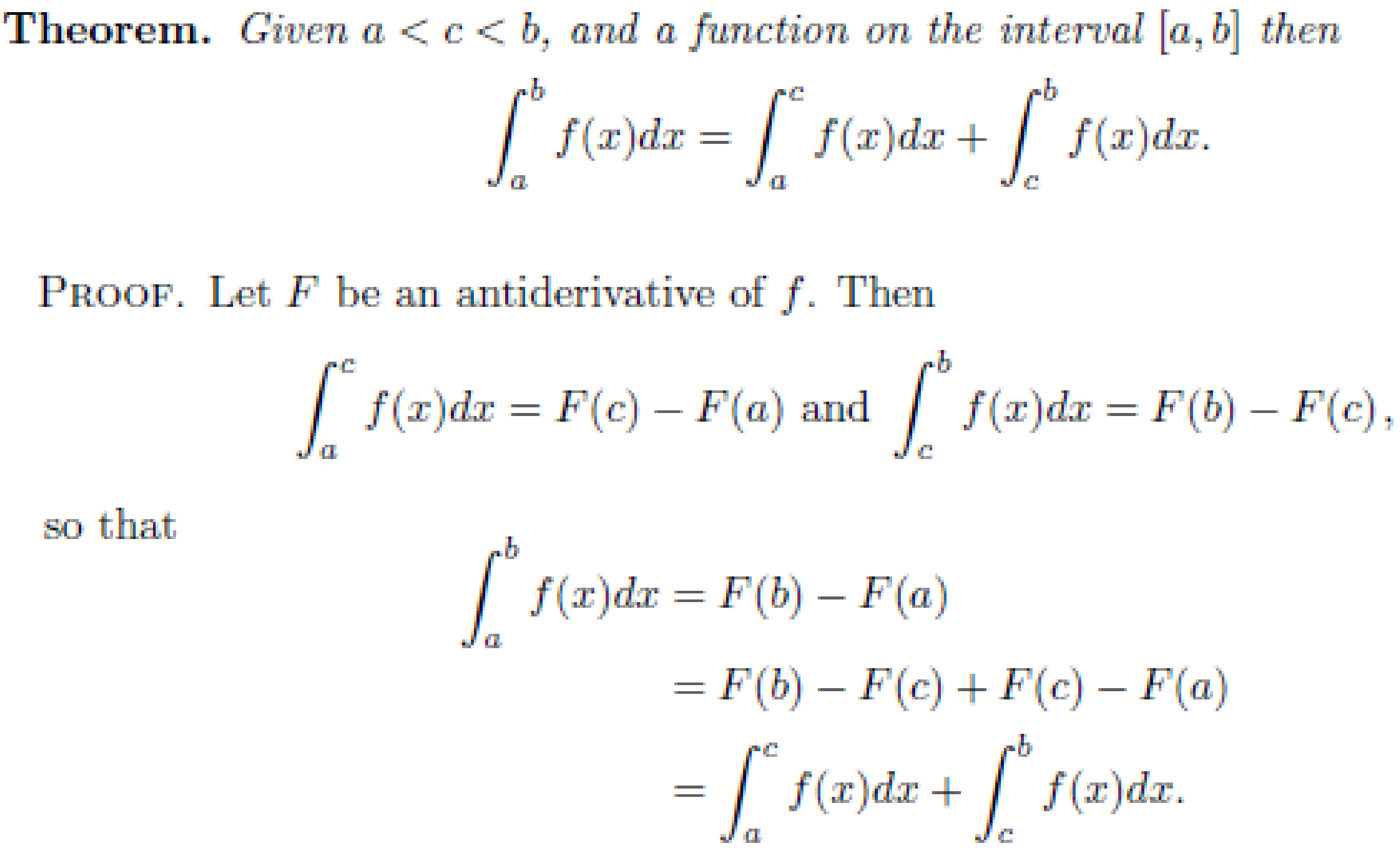
Definite integrals – some properties

- splitting the interval of integration <a,b>
- change of bounds
- identical bounds
- even and odd functions (definite integrals of them)
- so called “mean value theorem”
- integral as a function of its integration bounds

Definite integrals – some properties (splitting the interval)

Definite integrals have one other property for which there is no analog in indefinite integrals: if you split the interval of integration into two parts, then the integral over the whole is the sum of the integrals over the parts.

The following theorem says it more precisely:



Theorem. *Given $a < c < b$, and a function on the interval $[a, b]$ then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

PROOF. Let F be an antiderivative of f. Then

$$\int_a^c f(x)dx = F(c) - F(a) \text{ and } \int_c^b f(x)dx = F(b) - F(c),$$

so that

$$\begin{aligned} \int_a^b f(x)dx &= F(b) - F(a) \\ &= F(b) - F(c) + F(c) - F(a) \\ &= \int_a^c f(x)dx + \int_c^b f(x)dx. \end{aligned}$$

Definite integrals – some properties (change of bounds)

So far we have always assumed that a < b in all definite integrals $\int_a^b \ldots$.
The fundamental theorem suggests that when b < a, we should define the integral as:

$$\int_a^b f(x)dx = F(b) - F(a) = -(F(a) - F(b)) = -\int_b^a f(x)dx.$$

For instance,

$$\int_1^0 x dx = -\int_0^1 x dx = -\frac{1}{2}.$$

Simply: When we change the order of integral bounds, the final integral will change its sign.

Comment: Integral over the interval with length zero – is equal to zero:

$$\int_a^a f(x)\, dx = 0.$$

Definite integrals – some properties (even and odd functions)

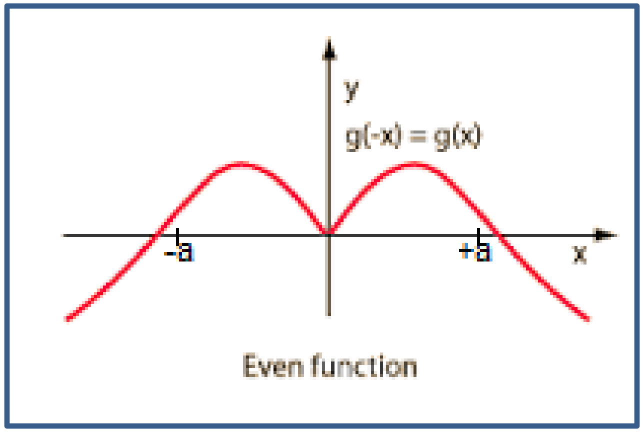
When we have an even function *g*(*x*) on a symmetrical interval <-a, a>, then it is valid:

$$\int_{-a}^{a} g(x)dx = \int_{-a}^{0} g(x)dx + \int_{0}^{a} g(x)dx = -\int_{0}^{-a} g(x)dx + \int_{0}^{a} g(x)dx =$$

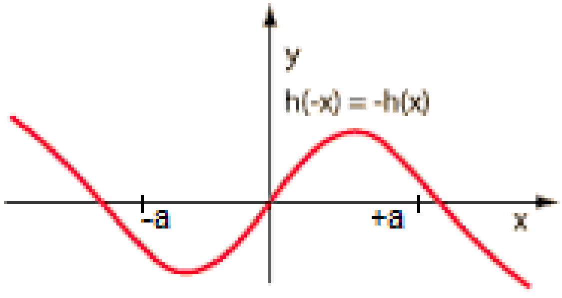
$$= -\int_{0}^{-a} g(x)dx \left[\begin{array}{c} t = -x \\ dt = -dx \\ x = -a,\ t = a \end{array} \right] + \int_{0}^{a} g(x)dx = \int_{0}^{a} g(-t)dt + \int_{0}^{a} g(x)dx =$$

$$= \int_{0}^{a} g(t)dt + \int_{0}^{a} g(x)dx = \int_{0}^{a} g(x)dx + \int_{0}^{a} g(x)dx = \underline{2\int_{0}^{a} g(x)dx}$$

The result is twice the area from the half of the interval.



Even function



Odd function

Definite integrals – some properties (even and odd functions)

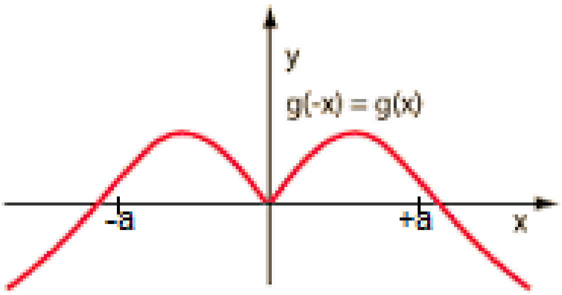
When we have an odd function *g*(*x*) on a symmetrical interval <-a, a>, then it is valid:

$$\int_{-a}^{a} h(x)dx = \int_{-a}^{0} h(x)dx + \int_{0}^{a} h(x)dx = -\int_{0}^{-a} h(x)dx + \int_{0}^{a} h(x)dx =$$

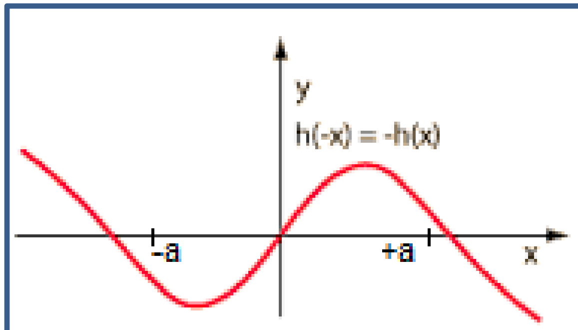
$$= -\int_{0}^{-a} h(x)dx \begin{bmatrix} t = -x \\ dt = -dx \\ x = -a,\ t = a \end{bmatrix} + \int_{0}^{a} h(x)dx = \int_{0}^{a} h(-t)dt + \int_{0}^{a} h(x)dx =$$

$$= \int_{0}^{a} -h(t)dt + \int_{0}^{a} h(x)dx = -\int_{0}^{a} h(x)dx + \int_{0}^{a} h(x)dx \underline{=0}$$

The result is zero.



Even function



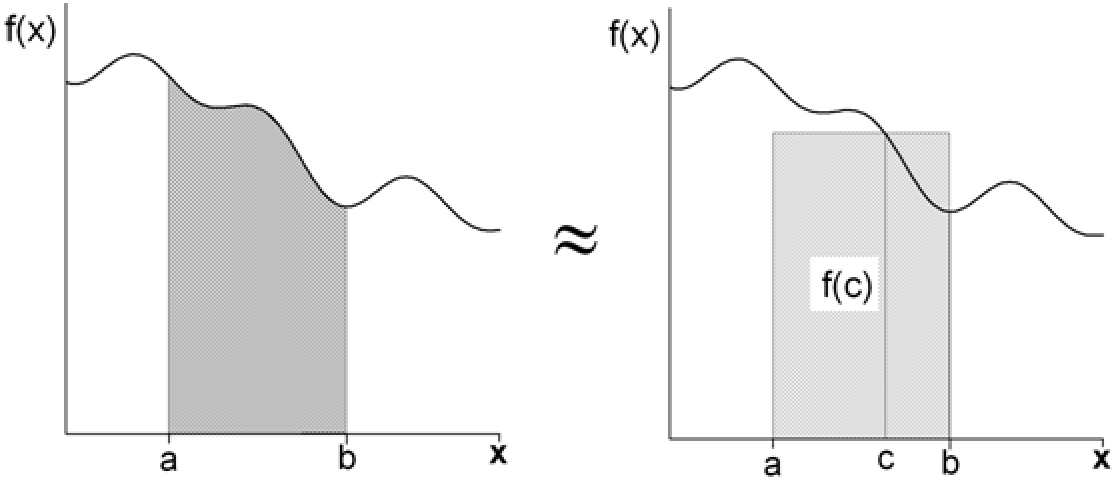
Odd function

Definite integrals – some properties (mean value theorem)

First mean value theorem for definite integrals:

Let *f*(*x*) be a continuous function on <*a*, *b*>. Then there exists a point *c* in <*a*, *b*>, such that it is valid:

$$\int_a^b f(x)\,dx \approx f(c)(b-a)$$



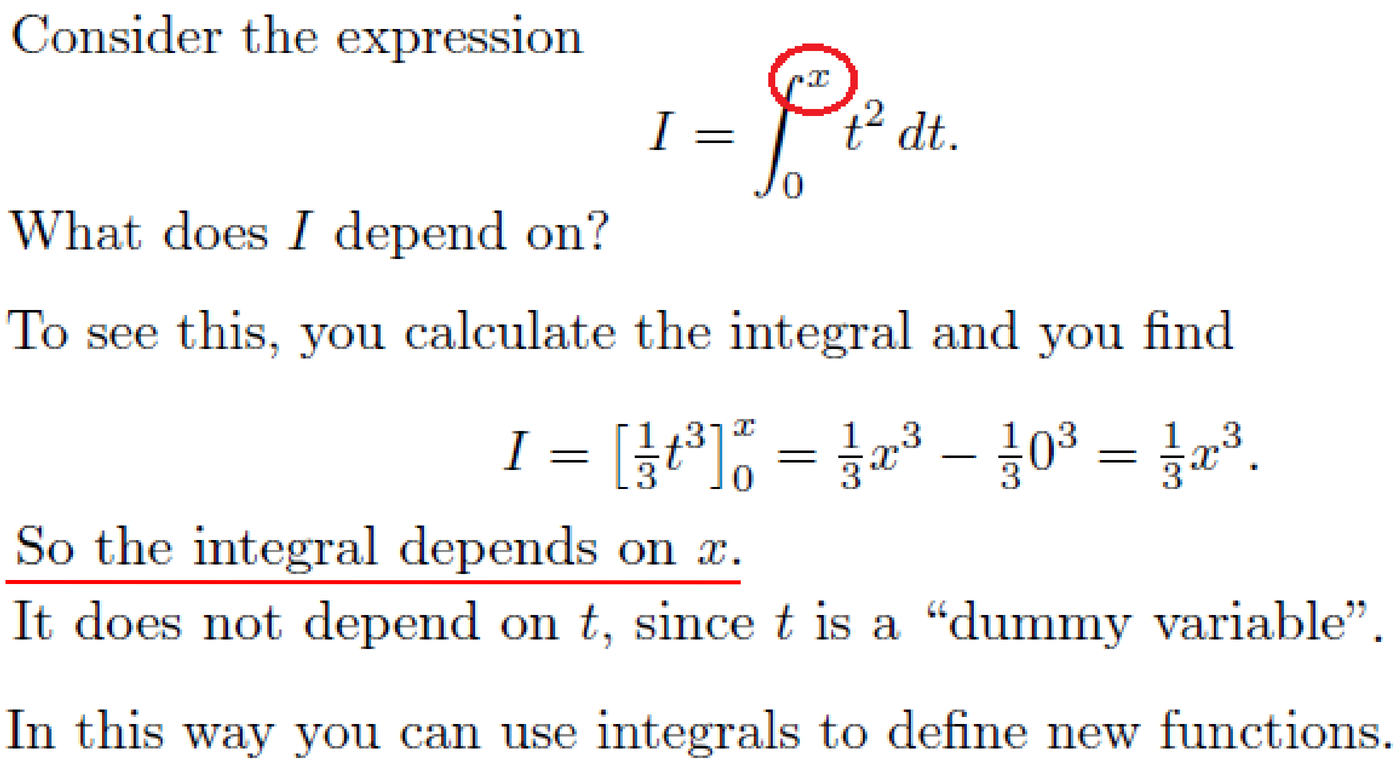
Simply: The area below the graph *f*(*x*) can be approximated by a rectangle with the height *f*(*c*) – and for continuous functions there always exists such a point *c* in <*a*, *b*>

(... and we do not have to know its exact value).

Definite integrals – some properties

(the definite integral as a function of its integration bounds)

What will happen when we set for the upper bound not a number, but a variable?



Consider the expression

$$I = \int_0^x t^2\,dt.$$

What does I depend on?

To see this, you calculate the integral and you find

$$I = \left[\tfrac{1}{3}t^3\right]_0^x = \tfrac{1}{3}x^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3}x^3.$$

So the integral depends on x.

It does not depend on t, since t is a “dummy variable”.

In this way you can use integrals to define new functions.

Such expressions are called as integrals as a function of its upper bound (in general: function as a function of its integration bounds).

The definite integral as a function of its integration bounds.

An example of a function defined by an integral is the “error-function” from statistics. It is given by

$$\operatorname{erf}(x) \overset{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

so $\operatorname{erf}(x)$ is the area of the shaded region in figure.

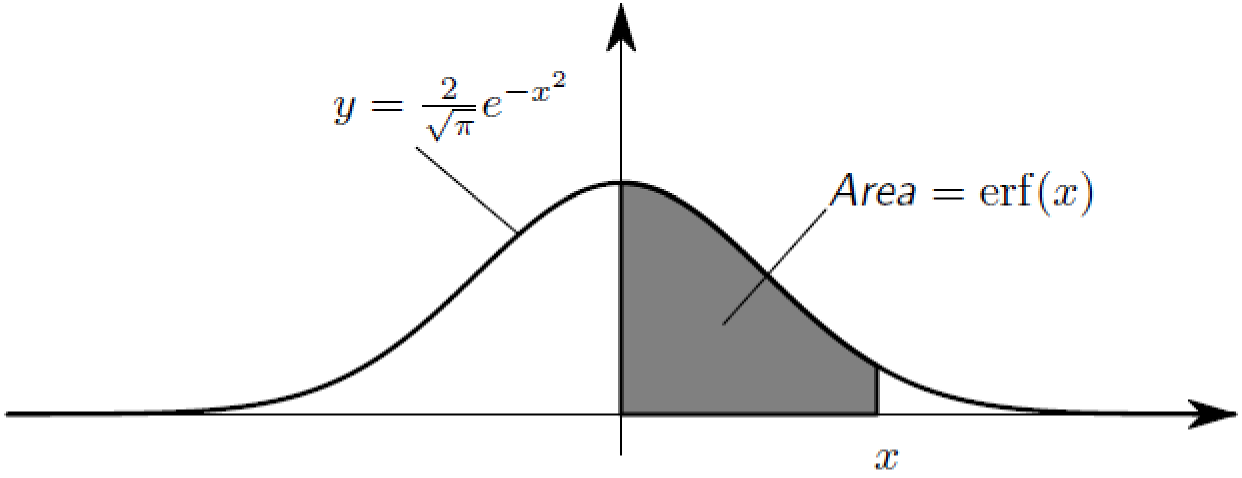


Figure. Definition of the Error function.

The integral cannot be computed in terms of the standard functions (square and higher roots, sine, cosine, exponential and logarithms).

Since the integral occurs very often in statistics it has been given a name, namely, “$\operatorname{erf}(x)$”.

The definite integral as a function of its integration bounds.

An interesting question arises (?):

How do you differentiate a function that is defined by an integral?

The answer is simple.

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\{F(x) - F(a)\} = F'(x) = f(x),$$

and therefore

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

Comment: For indefinite integrals it is clear that the differentiation of their result is the original function (it is an opposite operation). But this is not the case for definite integrals (because the results of them are numbers=constants); Only for the definite integral as a function of its integration bounds we can write this result.

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Definite integrals – improper integrals

The Riemann‘s definition of a definite integral requires the function f(x) to be bounded (smaller in absolute value than infinity).

We call a definite integral as **improper**, when:

a) function f(x) becomes unbounded on <a,b> or

b) the interval <a,b> becomes unbounded (i.e. a = –∞ and/or b = +∞).

Both situations can be sometimes solved by means of the solution of a limit case (when the solution converges):

case a):

$$\lim_{c\to a^+}\int_c^b f(x)\,\mathrm{d}x,\ \lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x$$

case b):

$$\lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x,\ \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

Example (case a)):

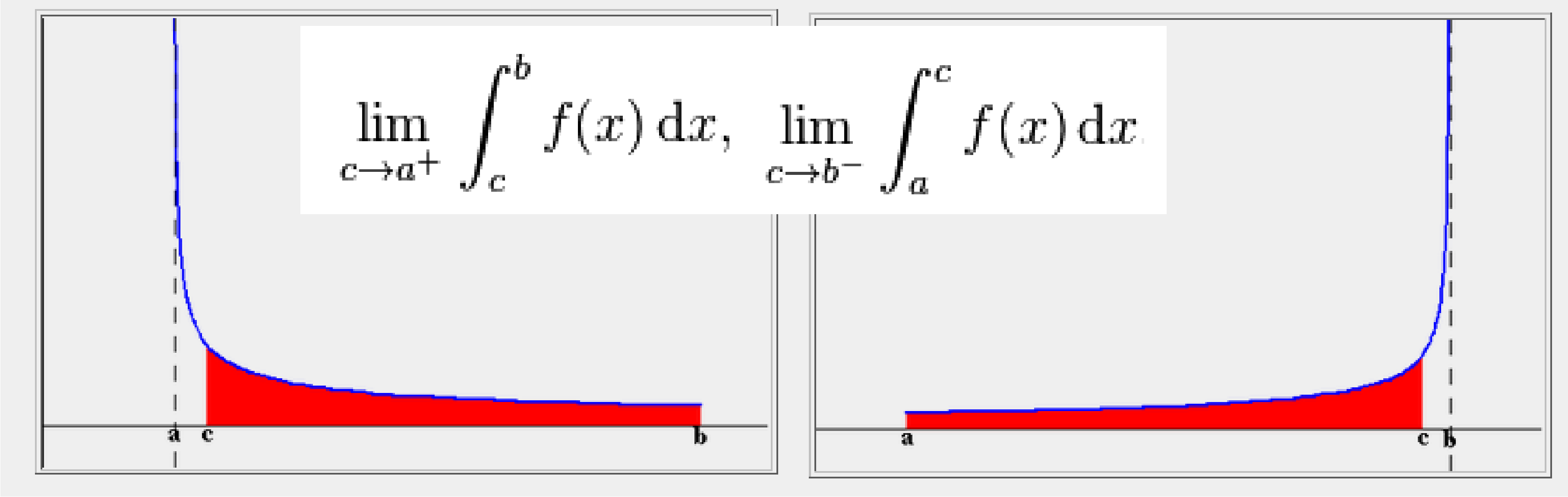
$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to0+}\int_a^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to0+}(2-2\sqrt{a}) = 2$$

Example (case b)):

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,\mathrm{d}x = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1}\right) = 1$$

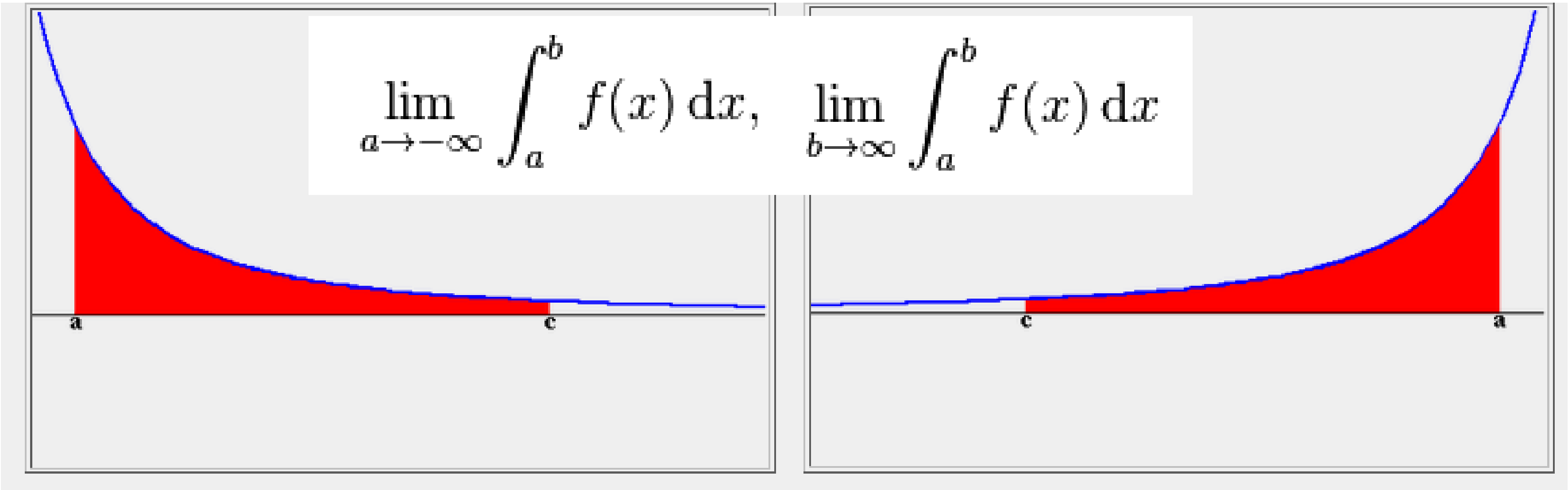
Definite integrals – improper integrals

case a):



$$\lim_{c\to a^+}\int_c^b f(x)\,\mathrm{d}x,\quad \lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x$$

case b):



$$\lim_{a\to -\infty}\int_a^b f(x)\,\mathrm{d}x,\quad \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

Definite integrals – improper integrals

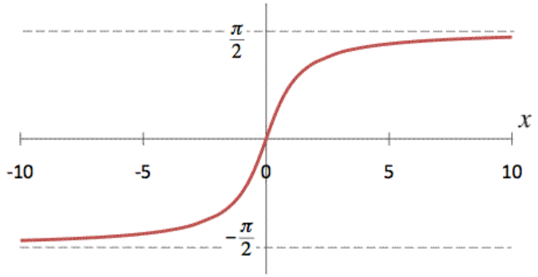
Example (case a)):

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{\sqrt[3]{x^2}} &= \lim_{s\to 0}\int_{-1}^{-s}\frac{dx}{\sqrt[3]{x^2}} + \lim_{t\to 0}\int_{t}^{1}\frac{dx}{\sqrt[3]{x^2}}\\ &= \lim_{s\to 0} 3(1-\sqrt[3]{s}) + \lim_{t\to 0} 3(1-\sqrt[3]{t})\\ &= 3+3\\ &= 6.\end{aligned}$$

Example (case b)):

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = 2\int_{0}^{+\infty}\frac{dx}{1+x^2} = 2\lim_{t\to\infty}\int_{0}^{t}\frac{dx}{1+x^2} = 2\lim_{t\to\infty}\left[arctan(x)\right]_0^t =$$

$$= 2\lim_{t\to\infty}\left[arctan(t) - arctan(0)\right] = 2\lim_{t\to\infty}\left[arctan(t)\right] = \pi$$



Definite integrals – improper integrals

Example (case b)) – sometimes a substitution can cause an occurrence of an improper integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}\,\mathrm{d}x = \left| \begin{array}{rcl} \operatorname{tg} x &=& t \\ x &=& \operatorname{arctg} t \\ \mathrm{d}x &=& \frac{1}{1+t^2}\,\mathrm{d}t \\ -\frac{\pi}{2} &\rightsquigarrow& -\infty \\ \frac{\pi}{2} &\rightsquigarrow& +\infty \end{array} \right| = \int_{-\infty}^{\infty} \frac{1}{1+\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{1}{1+2t^2}\,\mathrm{d}t =$$

$$= \int_{-\infty}^{\infty} \frac{1}{1+(\sqrt{2}\,t)^2}\,\mathrm{d}t = \frac{1}{\sqrt{2}}\left[\operatorname{arctg}\sqrt{2}\,t\right]_{-\infty}^{+\infty} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \frac{\sqrt{2}}{2}\pi.$$

Comment: Some integrals do not converge – in such cases they reach infinite values (they diverge).

Example:

$$\int_{1}^{\infty} \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty}\int_{1}^{b} \frac{1}{x}\,\mathrm{d}x = \infty.$$

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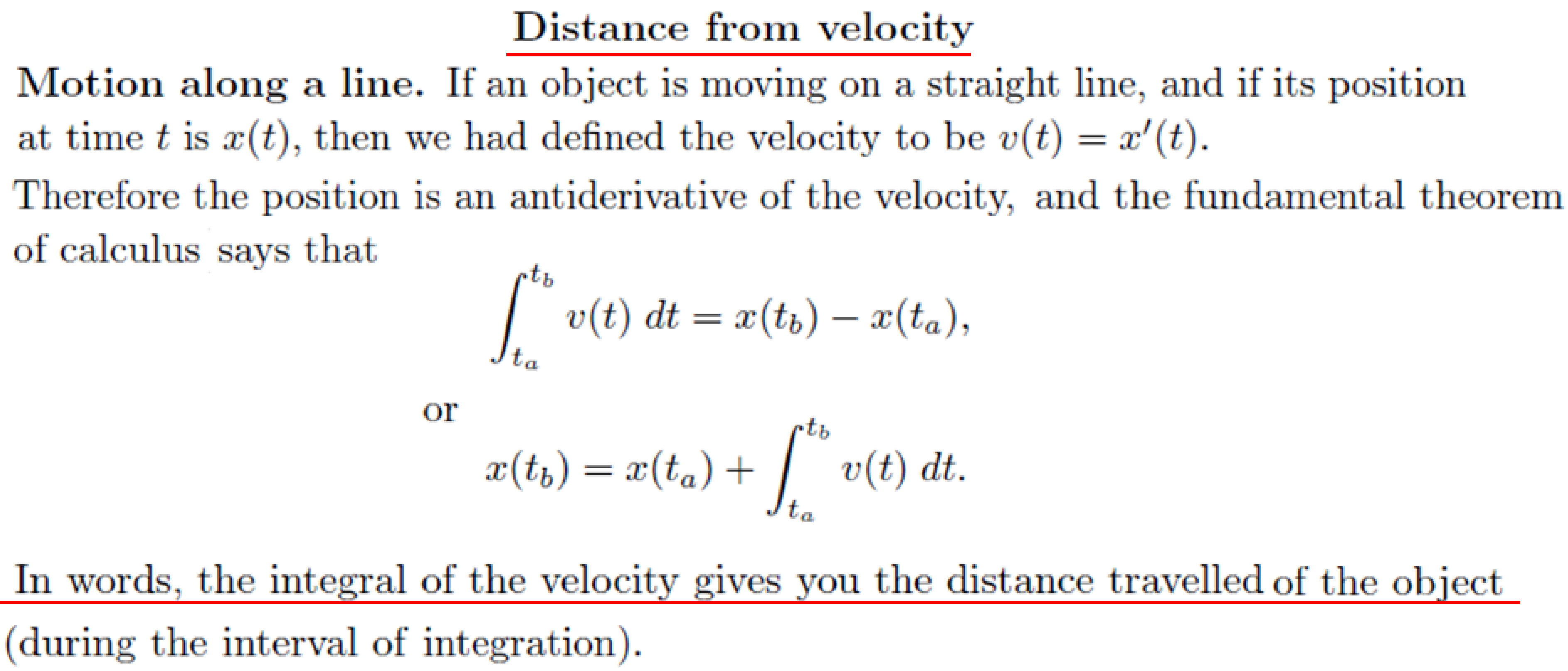
Definite integrals – applications

Many definite integrals have applications in mathematics and physics.

Let us give some of them.



Distance from velocity

Motion along a line. If an object is moving on a straight line, and if its position at time t is $x(t)$, then we had defined the velocity to be $v(t) = x'(t)$.

Therefore the position is an antiderivative of the velocity, and the fundamental theorem of calculus says that

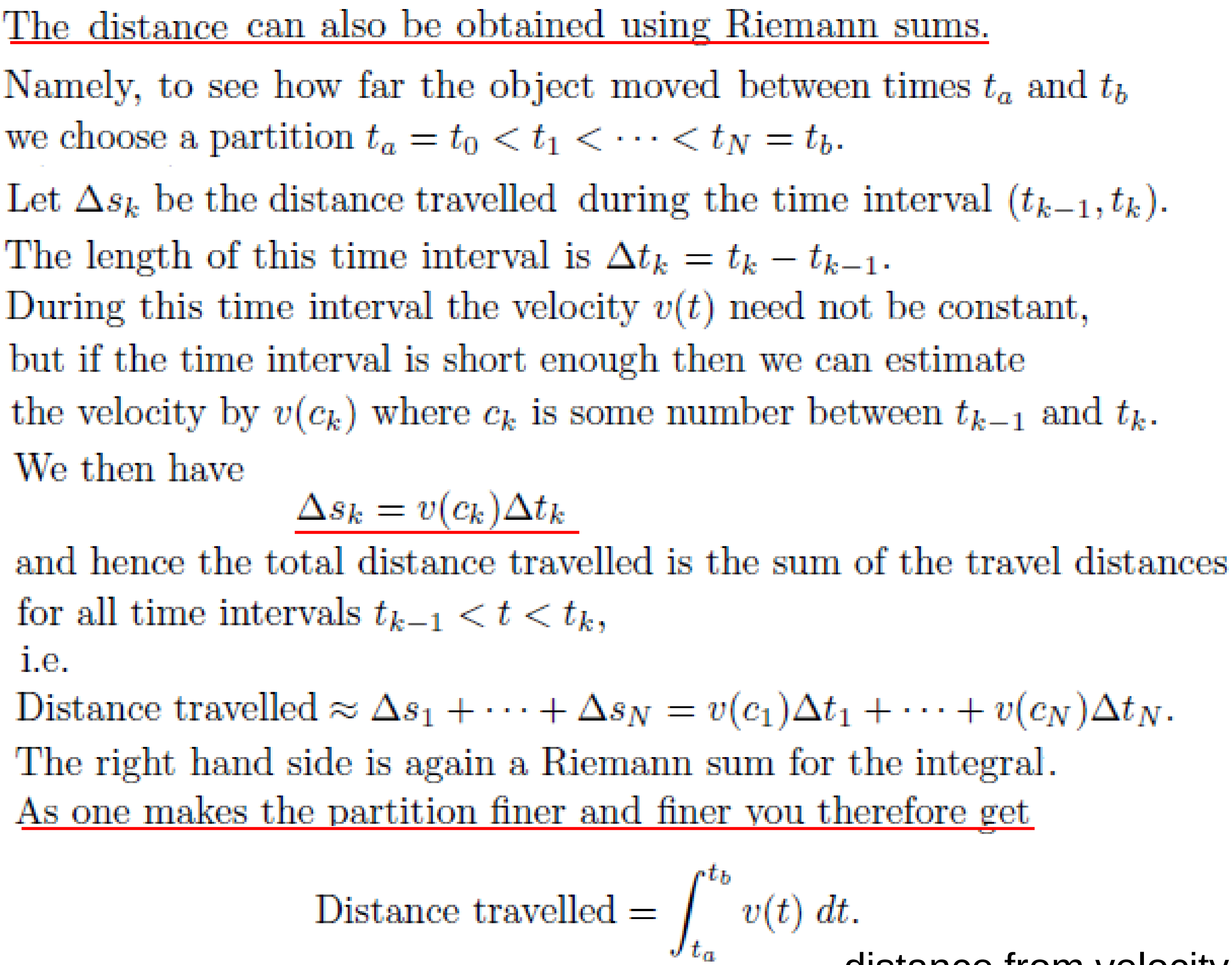
$$\int_{t_a}^{t_b} v(t)\, dt = x(t_b) - x(t_a),$$

or

$$x(t_b) = x(t_a) + \int_{t_a}^{t_b} v(t)\, dt.$$

In words, the integral of the velocity gives you the distance travelled of the object (during the interval of integration).

Different way of derivation – by means of geometrical meaning:



The distance can also be obtained using Riemann sums.

Namely, to see how far the object moved between times t_a and t_b we choose a partition $t_a = t_0 < t_1 < \cdots < t_N = t_b$.

Let Δs_k be the distance travelled during the time interval (t_{k-1}, t_k). The length of this time interval is $\Delta t_k = t_k - t_{k-1}$. During this time interval the velocity $v(t)$ need not be constant, but if the time interval is short enough then we can estimate the velocity by $v(c_k)$ where c_k is some number between t_{k-1} and t_k. We then have

$$\Delta s_k = v(c_k)\Delta t_k$$

and hence the total distance travelled is the sum of the travel distances for all time intervals $t_{k-1} < t < t_k$, i.e.

$$\text{Distance travelled} \approx \Delta s_1 + \cdots + \Delta s_N = v(c_1)\Delta t_1 + \cdots + v(c_N)\Delta t_N.$$

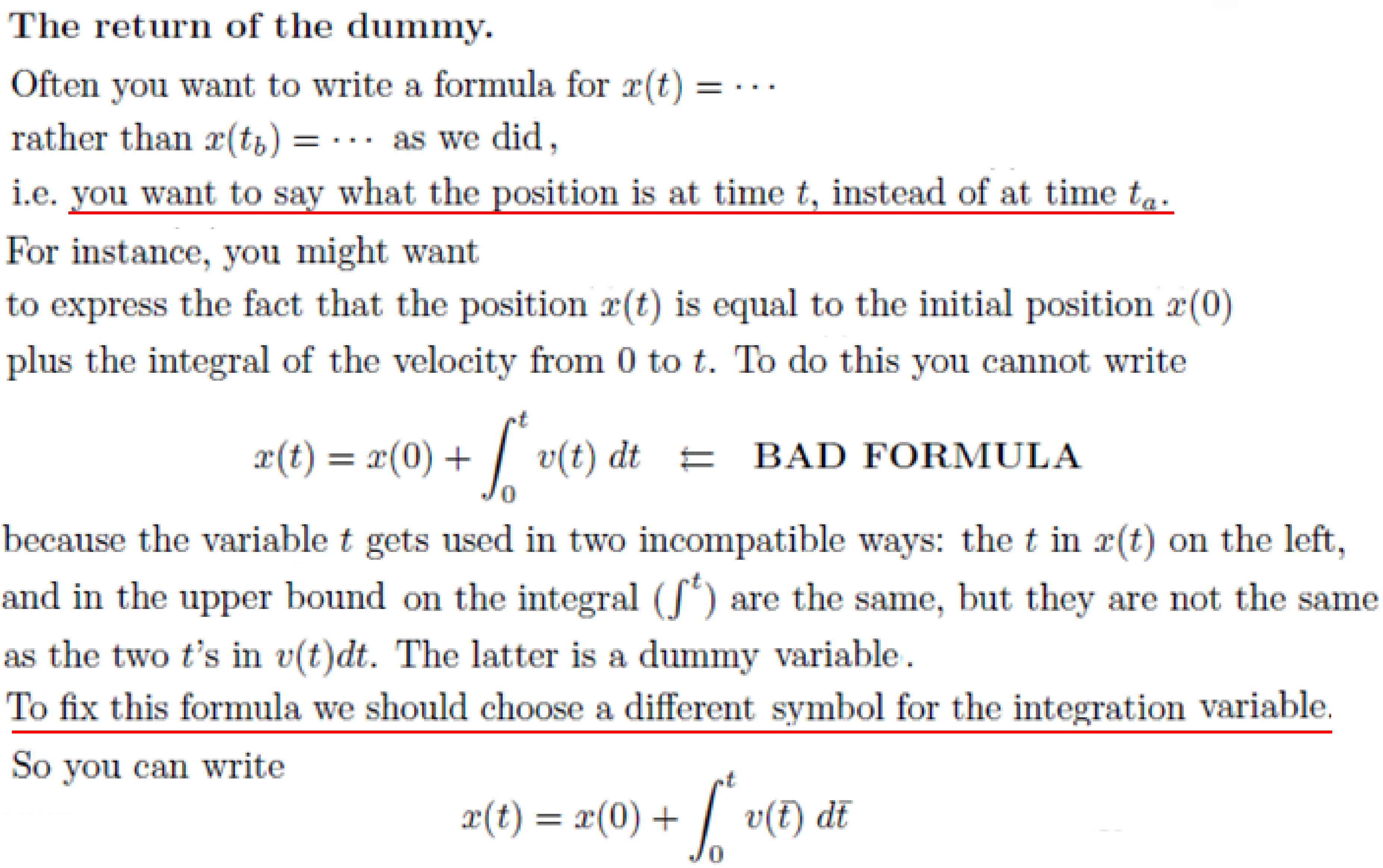
The right hand side is again a Riemann sum for the integral. As one makes the partition finer and finer you therefore get

$$\text{Distance travelled} = \int_{t_a}^{t_b} v(t)\,dt.$$

distance from velocity (2/3)

$$x(t_b) = x(t_a) + \int_{t_a}^{t_b} v(t)\, dt$$

additional comment:



The return of the dummy.

Often you want to write a formula for $x(t) = \cdots$
rather than $x(t_b) = \cdots$ as we did,
i.e. you want to say what the position is at time t, instead of at time t_a.

For instance, you might want
to express the fact that the position $x(t)$ is equal to the initial position $x(0)$
plus the integral of the velocity from 0 to t. To do this you cannot write

$$x(t) = x(0) + \int_0^t v(t)\, dt \quad \Leftarrow \quad \textbf{BAD FORMULA}$$

because the variable t gets used in two incompatible ways: the t in $x(t)$ on the left, and in the upper bound on the integral ($\int^t$) are the same, but they are not the same as the two t's in $v(t)dt$. The latter is a dummy variable.

To fix this formula we should choose a different symbol for the integration variable.

So you can write

$$x(t) = x(0) + \int_0^t v(\bar{t})\, d\bar{t}$$

Definite integrals – applications

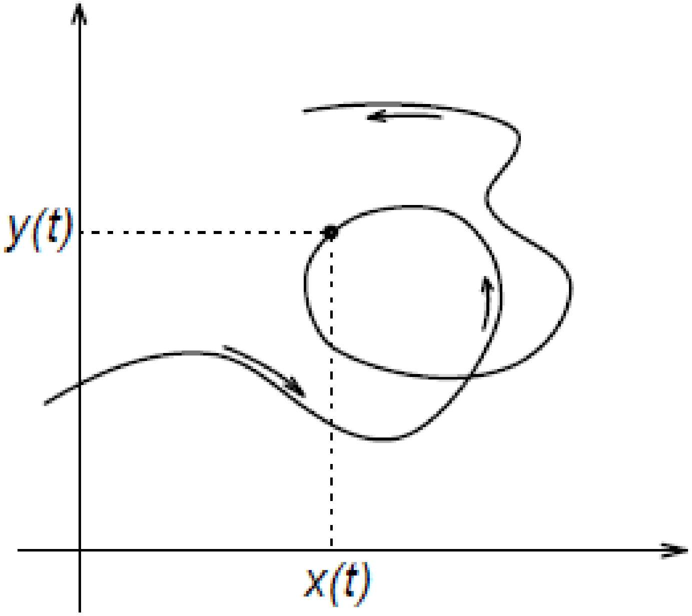
Next application:

Motion in the plane – parametric curves.

To describe the motion of an object in the plane you could keep track of its x and y coordinates at all times t.

This would give you *two* functions of t, namely, $x(t)$ and $y(t)$, both of which are defined on the same interval $t_0 \leq t \leq t_1$ which describes the duration of the motion you are describing.

In this context a pair of functions $(x(t), y(t))$ is called *a parametric curve.*



Definite integrals – applications

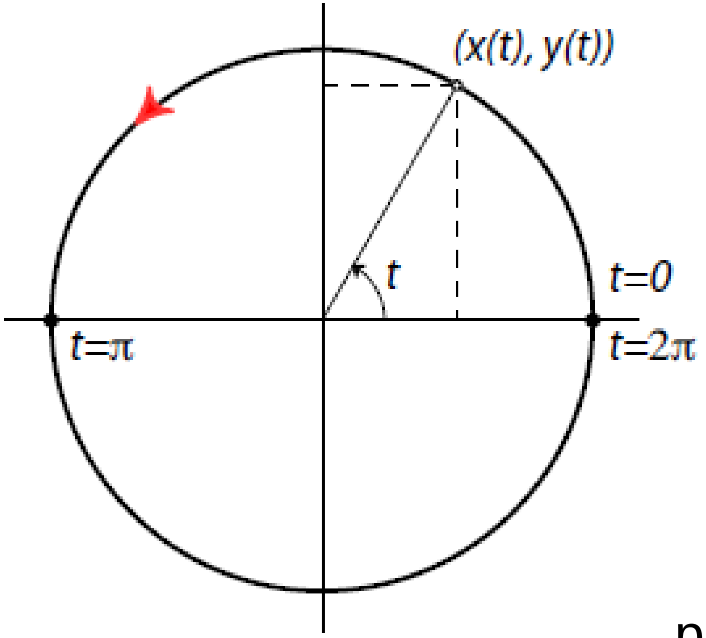
As an example, consider the motion described by

$$x(t) = \cos t, \quad y(t) = \sin t (0 \leq t \leq 2\pi).$$

In this motion the point $(x(t), y(t))$ lies on the unit circle since

$$x(t)^2 + y(t)^2 = \cos^2 t + \sin^2 t = 1.$$

As t increases from 0 to 2π the point $(x(t), y(t))$ goes around the unit circle exactly once, in the counter-clockwise direction.



Definite integrals – applications

Length of a parametric curve.

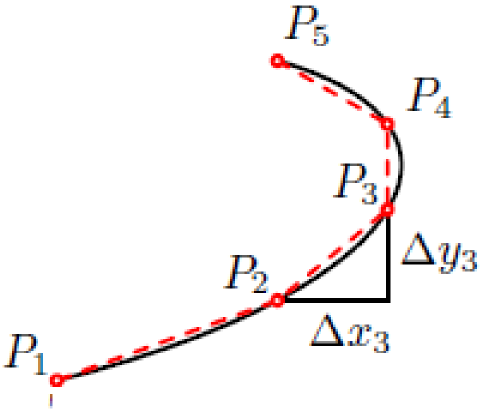
Let $(x(t), y(t))$ be some parametric curve defined for $t_a \leq t \leq t_b$.

Choose a partition $t_a = t_0 < t_1 < \cdots < t_N = t_b$ of the interval $[t_a, t_b]$.

You then get a sequence of points $P_0(x(t_0), y(t_0))$, $P_1(x(t_1), y(t_1))$, $\ldots$, $P_N(x(t_N), y(t_N))$, and after "connecting the dots" you get a polygon.

The distance between two consecutive points P_{k-1} and P_k is

$$\begin{aligned} \Delta s_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2}\ \Delta t_k \\ &\approx \sqrt{x'(c_k)^2 + y'(c_k)^2}\ \Delta t_k \end{aligned}$$



where we have approximated the difference quotients

$$\frac{\Delta x_k}{\Delta t_k} \text{ and } \frac{\Delta y_k}{\Delta t_k}$$

by the derivatives $x'(c_k)$ and $y'(c_k)$ for some c_k in the interval $[t_{k-1}, t_k]$.

Definite integrals – applications

The total length of the polygon is then

$$\sqrt{x'(c_1)^2 + y'(c_1)^2}\,\Delta t_1 + \cdots + \sqrt{x'(c_1)^2 + y'(c_1)^2}\,\Delta t_1$$

This is a Riemann sum for the integral $\int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt$, and hence we find (once more) that the length of the curve is

$$\boxed{s = \int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt.}$$

Example: the unit circle

Our length formula tells us that the length of the unit circle is

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

given $x(t) = \cos t, \quad y(t) = \sin t, \quad (0 \le t \le 2\pi)$

and computed $\sqrt{x'(t)^2 + y'(t)^2} = 1.$

Definite integrals – applications

The total length of the polygon is then

$$\sqrt{x'(c_1)^2 + y'(c_1)^2}\,\Delta t_1 + \cdots + \sqrt{x'(c_N)^2 + y'(c_N)^2}\,\Delta t_N$$

This is a Riemann sum for the integral $\int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt$, and hence we find (once more) that the length of the curve is

$$\boxed{s = \int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2}\,dt.}$$

The length of the graph of a function.

The graph of a function ($y = f(x)$ with $a \le x \le b$) is also a curve in the plane, and you can ask what its length is.

We will now find this length by *representing the graph as a parametric curve*.

The standard method of representing the graph of a function $y = f(x)$ by a parametric curve is to choose

$$x(t) = t, \text{ and } y(t) = f(t), \text{ for } a \le t \le b.$$

Since $x'(t) = 1$ and $y'(t) = f'(t)$ we find that the length of the graph is

$$\boxed{L = \int_a^b \sqrt{1 + f'(t)^2}\,dt.}$$

parametric curves -

- length of function (1/3)

Definite integrals – applications

The length of the graph of a function

The variable t in this integral is a dummy variable and we can replace it with any other variable we like, for instance, x:

$$\boxed{L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.}$$

Example:

We have a linear function *f*(*x*) = *x* and would like to know its length on the interval <0,2>:

$$L = \int_0^2 \sqrt{1+(x')^2}\,dx = \int_0^2 \sqrt{1+1}dx = \sqrt{2}\int_0^2 dx = \sqrt{2}[2-0] = 2\sqrt{2}$$

Can we somehow check this result on the graph?

Definite integrals – applications

The length of the graph of a function

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Example: Find the length of the function $f(x) = x^2/4 - \ln x/2$ on the interval <1,10>:

First of all, we need to express the derivative of this function:

$$\left(\frac{x^2}{4} - \frac{ln\,x}{2}\right)' = \frac{2x}{4} - \frac{1}{2}\frac{1}{x} = \frac{x}{2} - \frac{1}{2x} = \frac{x^2 - 1}{2x}$$

... and its squared version (plus one):

$$1 + \left[\frac{x^2 - 1}{2x}\right]^2 = 1 + \frac{x^4 - 2x^2 + 1}{4x^2} = \frac{4x^2 + x^4 - 2x^2 + 1}{4x^2} = \frac{x^4 + 2x^2 + 1}{4x^2} = \frac{\left(x^2 + 1\right)^2}{4x^2} = \left[\frac{x^2 + 1}{2x}\right]^2$$

Finally we can solve the integral:

$$L = \int_1^{10} \sqrt{\left[\frac{x^2 + 1}{2x}\right]^2}\, dx = \int_1^{10} \frac{x^2 + 1}{2x}\, dx = \frac{1}{2}\int_1^{10} x\,dx + \frac{1}{2}\int_1^{10} \frac{1}{x}\, dx = \frac{1}{4}\left[10^2 - 1\right] + \frac{1}{2}\left[ln10 - ln1\right] = \frac{99}{4} - \frac{ln10}{2}$$

parametric curves -

- length of function (3/3)