Mathematics for Biochemistry

LECTURE 3

Equations, Inequalities

Content

- Linear equations and inequalities
- Quadratic equations and inequalities
- Irrational equations and inequalities
- Exponential equations and inequalities
- Logarithmic equations and inequalities
- Trigonometric equations and inequalities

Equations and Inequalities

Available operations: adding/subtracting the same quantity to/from both sides

multiplication of both sides with non-zero constant

1. Domain of definition
2. Solving – available operations, properties of the functions
3. Result
4. Check

Linear equation → solution: **number(s)** (or function of parameter), **sets** or **none**

Linear inequality → solution: **interval** (or function of parameter) or **empty set**

Example 1.

$$10x - 1 = 15 - 6x \qquad \rightarrow \qquad D(f) = \mathbb{R}$$

$$10x + 6x = 15 + 1$$

$$16x = 16 \quad \Big| \cdot \frac{1}{16}$$

$$\underline{\underline{x = 1}} \qquad \rightarrow x \in D(f)$$

$$10 \cdot 1 - 1 = 15 - 6 \cdot 1$$

$$9 = 9$$

Example 2.

$$|4-2x|=12 \quad \rightarrow \quad D(f)=\mathbb{R}$$

Removing of absolute value results into two separate equations:

$$4-2x=12 \quad \vee \quad -(4-2x)=12$$

$$x=-4 \quad \vee \quad x=8$$

$$\underline{\underline{P=\{-4,8\}}}$$

Example 3.

$$|x-2| \le |x+4| \qquad \rightarrow \qquad D(f) = \mathbb{R}$$

Removing of abs. values results into 3 separate equations for each of intervals:

$a)\ (-\infty, -4)$

$-(x-2) \le -(x+4)$

$2 \le -4 \quad \Rightarrow P_a = \{\ \}$

$b)\ \langle -4, 2)$

$-(x-2) \le x+4$

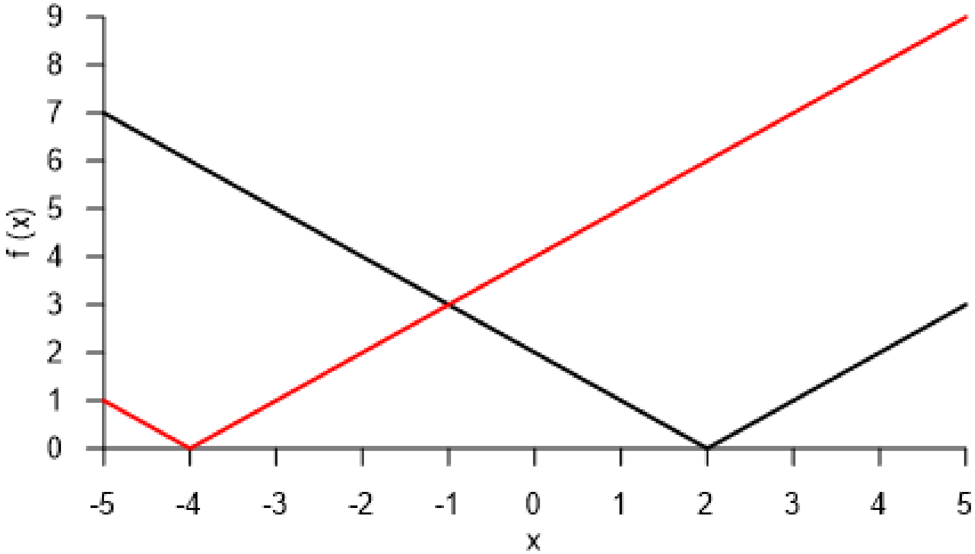
$x \ge -1 \quad \Rightarrow P_b = \langle -1, 2)$

$c)\ \langle 2, \infty)$

$x-2 \le x+4$

$-2 \le 4 \quad \Rightarrow P_c = \langle 2, \infty)$

$$P = P_a \cup P_b \cup P_c = \underline{\underline{\langle -1, \infty)}}$$



Quadratic equation → 2 solutions: **real** or **complex numbers**

Quadratic inequality → solution: **sets** of **real** or **complex numbers**

Example 4.

$$\frac{x+3}{x-3}+\frac{x+6}{x-6}=\frac{11}{5} \quad \rightarrow \quad D(f)=\mathbb{R}-\{3,6\}$$

$$x^2+33x-198=0 \quad \rightarrow \quad \begin{array}{l} x_1=-42 \\ x_2=9 \end{array}$$

$$P=\{-42,9\}$$

Example 5.

$$16 - 7x^2 = 79 \quad \rightarrow \quad D(f) = \mathbb{R}$$

$$7x^2 + 63 = 0$$

$$x^2 + 9 = 0$$

$$D = b^2 - 4ac = 0 - 4 \cdot 1 \cdot 9 < 0$$

$$\underline{\underline{P = \{\ \}}} \quad \rightarrow P = \{-3i, 3i\}$$

Example 6.

$$x^2 - 3|x+1| - x = 0 \qquad \rightarrow \qquad D(f) = \mathbb{R}$$

$a)\quad (-\infty, -1)$

$$x^2 - 3(-1)(x+1) - x = 0$$

$$D < 0 \quad \Rightarrow P_a = \{\ \}$$

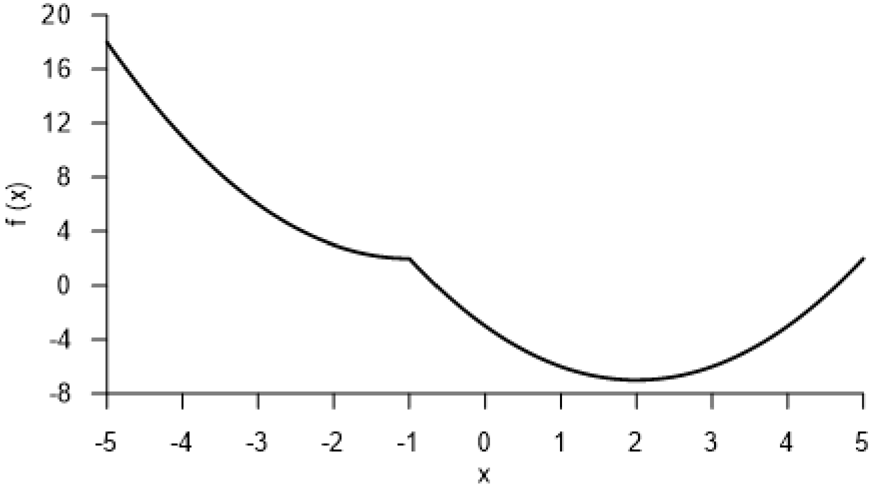
$b)\quad \langle -1, \infty)$

$$x^2 - 3(x+1) - x = 0$$

$$x^2 - 3x - 3 = 0$$

$$P_b = \langle 2-\sqrt{7}, 2+\sqrt{7} \rangle$$

$$P = P_a \cup P_b = \underline{\underline{\langle 2-\sqrt{7}, 2+\sqrt{7} \rangle}}$$



Irrational equation
Irrational inequality

Example 7.

$$\sqrt{1+x\sqrt{x^2+24}}=x+1$$

(for now, we will not calculate the D(f), however, try to find it later)

$$\sqrt{1+x\sqrt{x^2+24}}=x+1 \quad / \quad x+1\geq 0 \quad /^2$$

$$1+x\sqrt{x^2+24}=(x+1)^2$$

$$x\sqrt{x^2+24}=x(x+2) \quad \Rightarrow \quad x_1=0$$

$$\sqrt{x^2+24}=x+2 \,/^2$$

$$4x-20=0 \quad \Rightarrow \quad x_2=5$$

$$P=\underline{\underline{\{0,5\}}}$$

Irrational equation (inequality) – always check the results

Irrational equation
Irrational inequality

Example 8.

$$\sqrt{6x-x^2} < x+3 \qquad \to \qquad D(f)=\langle 0,6\rangle$$

$$\sqrt{6x-x^2} < x+3 \quad / \quad x+3 \geq 0 \quad /^2$$

$$6x-x^2 < (x+3)^2$$

$$x^2 > -\frac{9}{2} \quad \Rightarrow \text{ valid } \forall x \in D(f) \Rightarrow P = D(f)$$

$$P = \underline{\underline{\langle 0,6\rangle}}$$

Irrational equation
Irrational inequality

Example 9.

$$\sqrt{4-x^2}+\frac{|x|}{x}\geq 0 \qquad \rightarrow \qquad D(f)=\langle -2,2\rangle-\{0\}$$

a) for $x\in\langle -2,0)$ $\rightarrow \sqrt{4-x^2}-1\geq 0$ $\rightarrow x^2\leq 3$ $\rightarrow P_a=\langle -2,0)\cap\langle -\sqrt{3},\sqrt{3}\rangle=\langle -\sqrt{3},0)$

b) for $x\in(0,2\rangle$ $\rightarrow \sqrt{4-x^2}+1\geq 0$ $\rightarrow \forall x\in(0,2\rangle$ $\rightarrow P_b=(0,2\rangle$

$$P=P_a\cup P_b=\underline{\underline{\langle -\sqrt{3},2\rangle-\{0\}}}$$

Exponential equation
Exponential inequality

Example 9.

$$2^{3-x} = 4^{2-x} \quad \rightarrow \quad D(f) = \mathbb{R}$$

$$2^{3-x} = 4^{2-x}$$

$$2^{3-x} = \left(2^2\right)^{2-x}$$

$$2^{3-x} = 2^{4-2x} \quad \Rightarrow 3 - x = 4 - 2x \quad \rightarrow \quad x = 1$$

$$P = \underline{\underline{\{1\}}}$$

Exponential equation
Exponential inequality

Example 10.

$$x^2 2^{x+1} + 2^{|x-3|+2} = x^2 2^{|x-3|+4} + 2^{x-1} \qquad \rightarrow \qquad D(f) = \mathbb{R}$$

a) for $x \in (-\infty, 3\rangle \rightarrow x^2 2^{x+1} + 2^{-(x-3)+2} = x^2 2^{-(x-3)+4} + 2^{x-1}$

$$x^2 2^{x+1} + 2^{-x+5} = x^2 2^{-x+7} + 2^{x-1} \quad / \cdot 2^x$$

$$2x^2 2^{2x} + 2^5 = 2^7 x^2 + \frac{2^{2x}}{2}$$

$$2^{x-1}\left(4x^2 - 1\right) = 2^5\left(4x^2 - 1\right) \quad \rightarrow \quad 4x^2 - 1 = 0 \Rightarrow x_{1,2} = \pm\frac{1}{2}$$

$$2^{x-1} = 2^5 \Rightarrow x_3 = 3$$

$$P_a = \left\{-\frac{1}{2}, \frac{1}{2}, 3\right\}$$

b) for $x \in (3, \infty) \quad \rightarrow \quad x^2 2^{x+1} + 2^{x-3+2} = x^2 2^{x-3+4} + 2^{x-1}$

$$x^2 2^{x+1} + 2^{x-1} = x^2 2^{x+1} + 2^{x-1} \quad \Rightarrow \quad \text{valid } \forall \text{x} \in (3, \infty)$$

$$P_b = (3, \infty)$$

$$P = P_a \cup P_b = \underline{\underline{\left\{-\frac{1}{2}, \frac{1}{2}\right\} \cup \langle 3, \infty)}}$$

Exponential equation
Exponential inequality

Example 11.

$$4^x - 2 \cdot 5^{2x} - 10^x > 0 \qquad \rightarrow \qquad D(f) = \mathbb{R}$$

$$4^x - 2 \cdot 5^{2x} - 10^x > 0$$

$$2^{2x} - 2 \cdot 5^{2x} - 2^x 5^x > 0 \quad / \cdot \frac{1}{2^{2x}}$$

$$1 - 2 \cdot \left(\frac{5}{2}\right)^{2x} - \left(\frac{5}{2}\right)^x > 0 \quad / \left(\frac{5}{2}\right)^x = t$$

$$-2 \cdot t^2 - t + 1 > 0 \quad \Rightarrow t_1 = -1; \quad t_2 = \frac{1}{2}$$

Backward substitution:

$\left(\frac{5}{2}\right)^x = t_1 = -1$ This is not possible, so the interval for variable t “shrinks” to: $t \in \left(0, \frac{1}{2}\right)$

$$\left(\frac{5}{2}\right)^x = t_2 = \frac{1}{2} \Rightarrow x = \log_{\frac{5}{2}} \frac{1}{2} \qquad P = \underline{\underline{\left(-\infty, \log_{\frac{5}{2}} \frac{1}{2}\right)}}$$

Logarithmic equation
Logarithmic inequality

Example 12.

$$\log_x 4 + \log_x 2 = 1 \qquad \rightarrow \qquad D(f) = (0, \infty) - \{1\}$$

$$\log_x (4 \cdot 2) = 1$$

$$x^1 = 8 \Rightarrow x = 8$$

$$P = \underline{\underline{\{8\}}}$$

Logarithmic equation
Logarithmic inequality

Example 13.

$$\log_4(x+12)\cdot\log_x 2=1 \quad\rightarrow\quad D(f)=(0,\infty)-\{1\}$$

$$\log_4(x+12)\cdot\log_x 2=1 \quad / \log_a x=\frac{1}{\log_x a}$$

$$\log_4(x+12)\cdot\frac{1}{\log_2 x}=1$$

$$\log_4(x+12)=\log_2 x \quad / \log_a x=\frac{\log_b x}{\log_b a}$$

$$\log_4(x+12)=\frac{\log_4 x}{\boxed{\log_4 2}=\frac{1}{2}} \rightarrow \log_4(x+12)=2\log_4 x \quad / c\log_a x=\log_a x^c$$

$$\log_4(x+12)=\log_4 x^2 \Rightarrow x+12=x^2 \Rightarrow x_1=-3;\quad x_2=4$$

$$x_1\notin D(f)\Rightarrow P=\underline{\underline{\{4\}}}$$

Logarithmic equation
Logarithmic inequality

Example 14.

$$\log_{\frac{1}{2}}\left(\log_8 \frac{x^2-2x}{x-3}\right) > 0 \qquad \rightarrow \qquad D(f) = (3,\infty)$$

$$\log_{\frac{1}{2}}\left(\log_8 \frac{x^2-2x}{x-3}\right) > 0 \rightarrow \log_8 \frac{x^2-2x}{x-3} > 1 \rightarrow \frac{x^2-2x}{x-3} > 8$$

$$\frac{x^2-2x}{x-3} > 8 \rightarrow x^2 - 10x + 24 > 0 \Rightarrow x \in (4,6)$$

$$P = \underline{\underline{(4,6)}}$$

Trigonometric equation
Trigonometric inequality

Example 15.

$$2\sin x=\sqrt{3}\tan x \qquad \to \qquad D(f)=\mathbb{R}-\{k\pi\}_{k\in Z}$$

$$2\sin x=\sqrt{3}\frac{\sin x}{\cos x} \quad \Rightarrow \quad x_1=\{k\pi\}_{k\in Z}$$

$$2\cos x=\sqrt{3}$$

$$\cos x=\frac{\sqrt{3}}{2}\Rightarrow x_2=\left\{-\frac{\pi}{6}+2k\pi\right\}_{k\in Z} \quad ; \quad x_3=\left\{\frac{\pi}{6}+2k\pi\right\}_{k\in Z}$$

$$P=\underline{\underline{\left\{k\pi,-\frac{\pi}{6}+2k\pi,\frac{\pi}{6}+2k\pi\right\}_{k\in Z}}}$$

Trigonometric equation
Trigonometric inequality

Example 16.

$$\tan x = -\frac{1}{\tan x} + 2 \qquad \rightarrow \qquad D(f) = \mathbb{R} - \left\{\frac{\pi}{2} + k\pi\right\}_{k\in Z} - \{k\pi\}_{k\in Z}$$

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 2$$

$$\frac{1}{\sin x \cos x} = 2$$

$$\sin 2x = 1 \Rightarrow x = \frac{\pi}{4} + k\pi$$

$$P = \left\{\frac{\pi}{4} + k\pi\right\}_{k\in Z}$$

Trigonometric equation
Trigonometric inequality

Example 17.

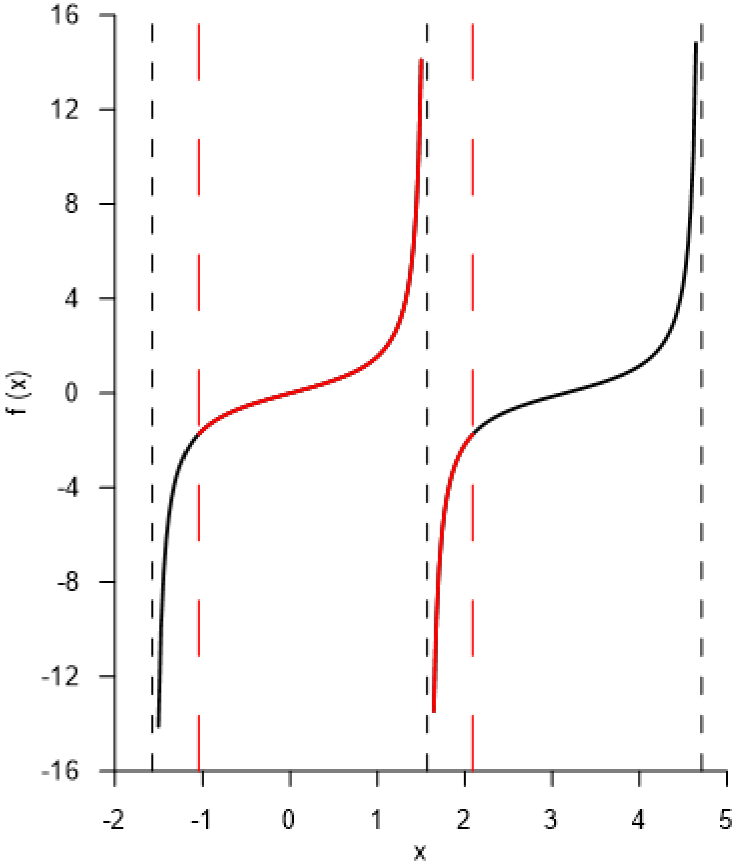
$$\sin x+\sqrt{3}\cos x>0 \qquad \rightarrow \qquad D(f)=\mathbb{R}$$

a) On interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\rightarrow\cos x>0\rightarrow\tan x>-\sqrt{3}\Rightarrow$

$$x\in\left(-\frac{\pi}{3},\frac{\pi}{2}\right)$$

b) On interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\rightarrow\cos x<0\rightarrow\tan x<-\sqrt{3}\Rightarrow$

$$x\in\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$$



$$P=\left(-\frac{\pi}{3},\frac{2\pi}{3}\right)+k\pi;\quad k\in Z$$