Mathematics for Biochemistry

LECTURE 5

Limits, Continuity

Lecture 5: limits of functions

Content:

- limit of a function
- methods of limits evaluation
- continuous function

Limit of a function - introduction

Limit of a function in some point speaks about special properties of a function and is very important in mathematical analysis.

Description (not a real definition):

If *f(x)* is some function then a limit of function *f* in point *a* is *L*:

$$\lim_{x \to a} f(x) = L$$

is to be read "the limit of *f(x)* as *x* approaches *a* is *L*".

(or in a very simple way "the limit of *f(x)* in *a* is *L* ")

It means that if we choose values of *x* which are close but not equal to *a*, then *f(x)* will be close to the value *L*;

moreover, *f(x)* gets closer and closer to *L* as *x* gets closer and closer to *a* (we can also say that f(x) converges to *L* for $x \to a$).

Comment: Point *a* can be also Infinity ($\pm\infty$).

Limit of a function - introduction

Example: If f(x) = x + 3 then

$$\lim_{x \to 4} (x+3) = 7$$

But this is a very simple example and for such situations we really do not need the whole concept of limits evaluation in mathematics.

We should inspect more special situations.

Example: If f(x) = sin(x)/x

$$\lim_{x \to 0} \frac{\sin(x)}{x} = ?$$

x	$\frac{\sin x}{x}$
1	0.841471...
0.1	0.998334...
0.01	0.999983...

This is not so a simple example, because when we substitute x=0 then we get a expression of 0/0 type, which does not exists.

But there is a solution and we will come to it (later on).

Limit of a function - introduction

Next example:

Unfortunately, substituting numbers can sometimes suggest a wrong answer.

..."*x* close to *a*„ – but how close is close enough?

Suppose we had taken the function:

$$\lim_{x \to 0} \frac{101\,000x}{100\,000x + 1} = ?$$

Substitution of some "small values of *x*" could lead us to believe that the limit is 1.
Only when we substitute very small values, we realize that the limit is 0 (zero)!

Limit of a function - introduction

$$\lim_{x \to 0} \frac{101000x}{100000x + 1} = 0$$

x	101000x/(100000x + 1)
1	1.0100
0.1	1.0099
0.01	1.0090
0.001	1.0000
0.0001	0.9182
0.00001	0.5050
10^{-6}	0.0918
10^{-7}	0.0100
10^{-8}	0.0010
10^{-9}	0.0001

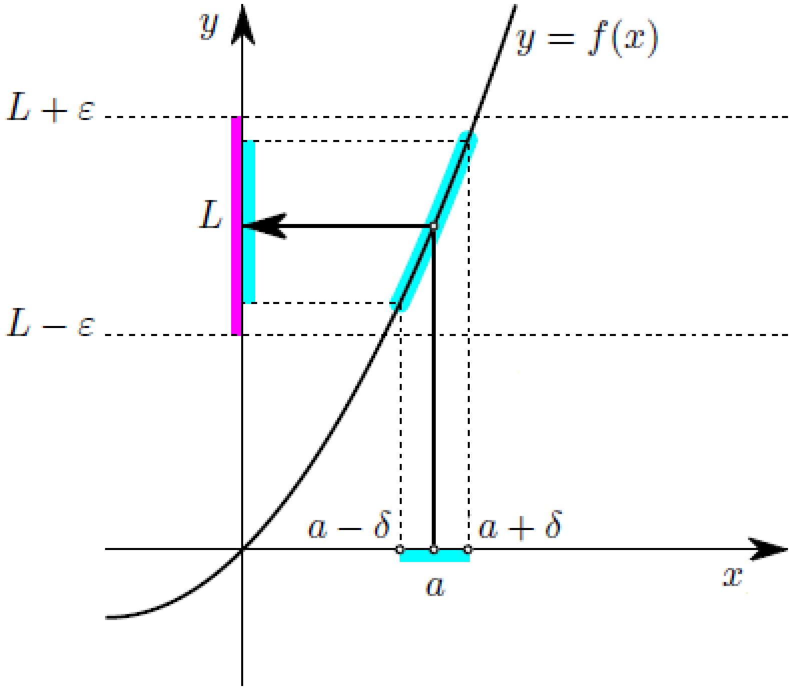
Limit of a function:

Definition: We say that L is the limit of *f(x)* as $x \to a$, if:

(1) *f(x)* need not be defined at *x* = *a*, but it must be defined for all other *x* in some interval which contains *a*.

(2) for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all *x* in the domain of *f(x)* one has:

$$|x - a| < \delta \ \textit{implies} \ |f(x) - L| < \varepsilon.$$



Limit of a function:

Definition: We say that L is the limit of *f(x)* as $x \to a$, if:

(1) *f(x)* need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.

(2) for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the domain of *f(x)* one has:

$$|x - a| < \delta \ \textit{implies} \ |f(x) - L| < \varepsilon.$$

Why the absolute values? The quantity $|x – a|$ is the distance between the points x and a on the number line, and one can measure how close x is to a by calculating $|x – a|$. The inequality $|x – a| < \delta$ says that "the distance between x and a is less than δ," or that "x and a are closer than δ."

Parameters δ and ε are also called as surroundings of points a and L, respectively.

Limit of a function: $|x - a| < \delta \ implies \ |f(x) - L| < \varepsilon.$

Evaluation of a limit, based on its definition.

Example: $$\lim_{x \to 5}(2x+1) = 11$$

Solution:

We have *f(x)* = 2*x* + 1, *a* = 5 and *L* = 11, and the question we must answer is: "how close should *x* be to 5 if want to be sure that *f(x)* = 2*x* + 1 differs less than ε from *L* = 11?"

To figure this out we try to get an idea of how big |*f(x)* – *L*| is:

|*f(x)* – *L*| = |(2*x* + 1) – 11| = |2*x* – 10| = 2|*x* – 5| = 2|*x* – *a*|.

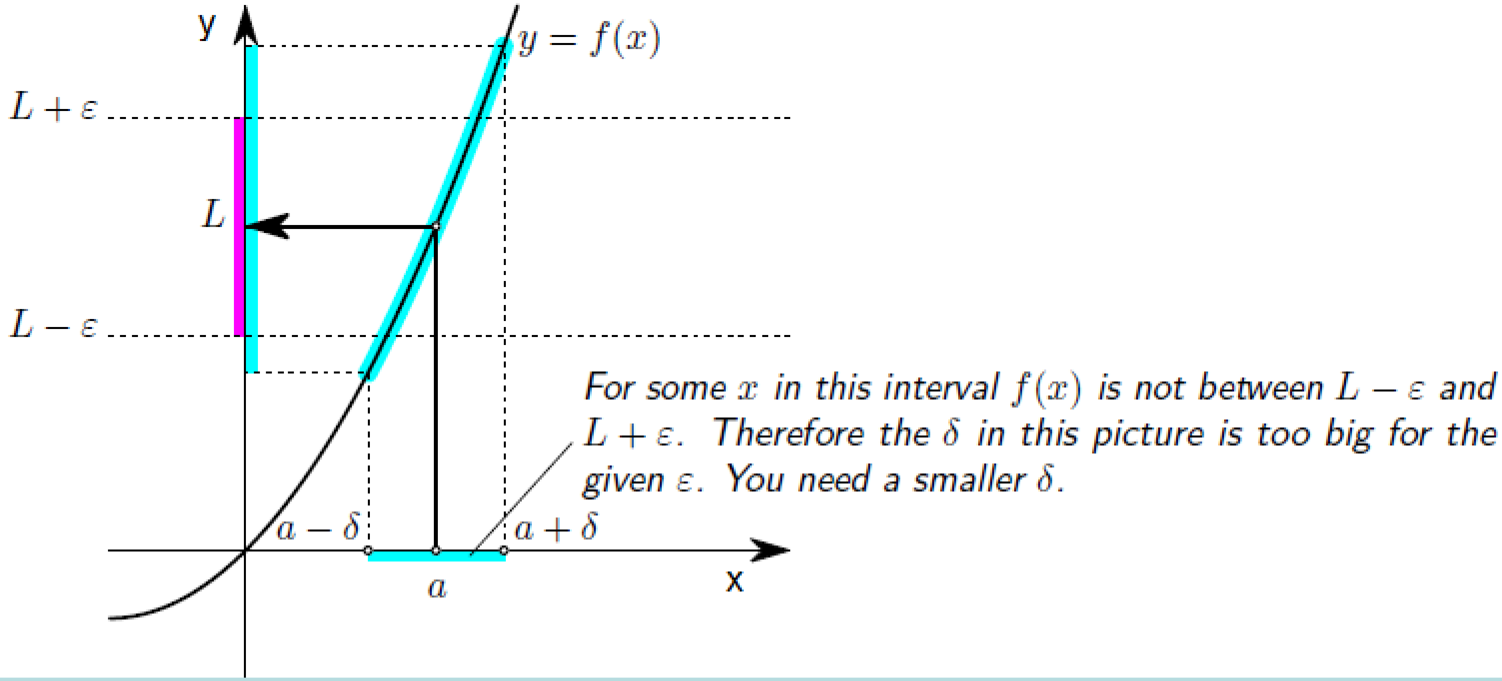
So, if 2|*x* – *a*| < ε then we have |*f(x)* – *L*| < ε, i.e.

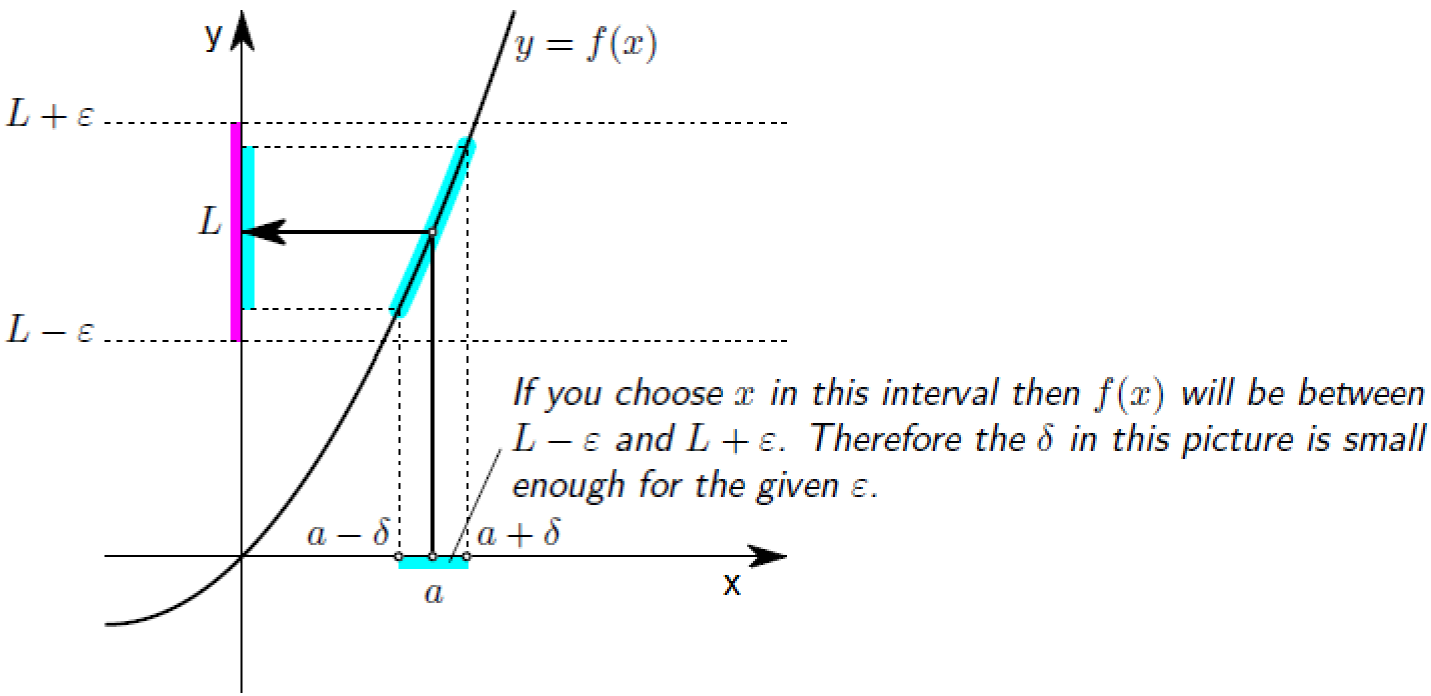
if |x – a| < 1/2ε then |f(x) – L| < ε.

We can therefore choose δ = 1/2ε. No matter what ε > 0 we are given our δ will also be positive, and if |x – 5| < δ then we can guarantee |(2x + 1) – 11| < ε.

That shows that $\lim_{x \to 5}$ (2x + 1) = 11.

This kind of solution is quite cumbersome, so we have to introduce some more efficient ways how to evaluate limits.


y
x
$y = f(x)$
$L + \varepsilon$
L
$L - \varepsilon$
$a - \delta$
$a + \delta$
a
For some x in this interval $f(x)$ is not between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is too big for the given ε. You need a smaller δ.


y
x
$y = f(x)$
$L + \varepsilon$
L
$L - \varepsilon$
$a - \delta$
$a + \delta$
a
If you choose x in this interval then $f(x)$ will be between $L - \varepsilon$ and $L + \varepsilon$. Therefore the δ in this picture is small enough for the given ε.

Methods of limits evaluation:

1. Substitution method
2. Factoring method
3. Conjugate method
4. Division method
5. L'Hospital's Rule

Comment: Rational function $f(x) = P_n(x)/Q_n(x)$, where $P_n(x)$ and $Q_n(x)$ are polynomials [$Q_n(x)$ is a nonzero polynomial].

$$\frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_0}$$

Methods of limits evaluation:

1. Substitution method:

Just simply put the value for *x* into the expression.

Simple examples:

$$\lim_{x \to 10} \frac{x}{2} = \frac{10}{2} = 5$$

$$\lim_{x \to -1} \frac{x^2 + 4x + 3}{x} = \frac{1 - 4 + 3}{-1} = \frac{0}{-1} = 0$$

But what to do in following cases?:

$$\lim_{x \to 3} \frac{x + 4x}{x - 3} = \frac{3 + 12}{0} = \frac{15}{0}$$

$$\lim_{x \to \infty} \frac{15}{x + 3} = \frac{15}{\infty}$$

Specific case:

What will happen when we must solve a limit, where we get finally an expressions of type 1/∞?

In fact 1/∞ is known to be undefined, because strictly speaking Infinity is not a number, it is an idea**.** But we can approach it.

x	$\frac{1}{x}$
1	1,00000
2	0,50000
4	0,25000
10	0,10000
100	0,01000
1.000	0,00100
10.000	0,00010

So - the limit of 1/x as x approaches Infinity is 0.

And what will happen when we take the exactly opposite case – expression of type 1/0?

Exactly the opposite situation (beside the fact that also this is undefined expression): The limit of 1/x as x approaches 0 is Infinity.

Next specific case:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

This is a so called indeterminate expression (form)

(these are expressions of type 0/0 or ∞/∞).

We will come to a general solution method for this kind of limits later on.

Methods of limits evaluation:

2. Factoring method:

Factoring – decomposition to factors, e.g.: $(x^2-1)=(x+1)(x-1)$

Example from previous slide:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}=\lim_{x\to 1}(x+1)=1+1=2$$

This method is mainly suitable for so called rational functions limits evaluation.

Next example:

$$\lim_{x\to -1}\frac{x^2+4x+3}{x+1}=?$$

Methods of limits evaluation:

3. Conjugate method:

Also for rational functions – sometime it helps, when we multiply the nominator and denominator of the fraction with a conjugate. Conjugate – in the case of binomials it is formed by negating the second term of the binomial (e.g. the conjugate of x+y is x-y).

Example:

$$\lim_{x\to 4}\frac{2-\sqrt{x}}{4-x}=\lim_{x\to 4}\frac{\left(2-\sqrt{x}\right)}{\left(4-x\right)}\frac{\left(2+\sqrt{x}\right)}{\left(2+\sqrt{x}\right)}=\lim_{x\to 4}\frac{\left(2^2+2\sqrt{x}-2\sqrt{x}-x\right)}{\left(4-x\right)\left(2+\sqrt{x}\right)}=$$

$$=\lim_{x\to 4}\frac{\left(4-x\right)}{\left(4-x\right)\left(2+\sqrt{x}\right)}=\lim_{x\to 4}\frac{1}{2+\sqrt{x}}=\frac{1}{2+\sqrt{4}}=\frac{1}{4}$$

Methods of limits evaluation:

4. Division method:

Valid only for limits of rational functions with $x \to \infty$.

$$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \lim_{x\to\infty}\frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_0}$$

1. option $\to \mathbf{n > m}$ $$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \infty$$

2. option $\to \mathbf{n < m}$ $$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = 0$$

3. option $\to \mathbf{n = m}$ $$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \frac{a_n}{b_m} \to$$ **Division of coefficient of the largest powers**

Methods of limits evaluation:

4. Division method:

Valid only for limits of rational functions with $x \to \infty$.

Solution is based on the division of all terms of both polynomials (in nominator and denominator) with the highest power of x.

Examples:

$$\lim_{x\to\infty}\frac{3x+1}{2x+5}=\lim_{x\to\infty}\frac{3x/x+1/x}{2x/x+5/x}=\lim_{x\to\infty}\frac{3+1/x}{2+5/x}=\frac{3+0}{2+0}=\frac{3}{2}$$

$$\lim_{x\to\infty}\frac{x^3+3x}{5x^3+2x^2+8}=\lim_{x\to\infty}\frac{x^3/x^3+3x/x^3}{5x^3/x^3+2x^2/x^3+8/x^3}=\lim_{x\to\infty}\frac{1+3/x^2}{5+2/x+8/x^3}=\frac{1}{5}$$

$$\lim_{x\to\infty}\frac{x^2}{x^3+1}=\lim_{x\to\infty}\frac{x^2/x^3}{x^3/x^3+1/x^3}=\lim_{x\to\infty}\frac{1/x}{1+1/x^3}=\frac{0}{1+0}=0$$

Methods of limits evaluation:

5. L'Hospital's Rule:

Valid for limits of so-called indeterminate expressions (forms) (expressions of type 0/0 or ∞/∞).

This rule is using derivatives, so we will return to it later during the term (future lectures).

Some special limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1;\ \lim_{x\to\infty}\frac{\sin x}{x}=0;\ \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e;\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e;\ \lim_{x\to 0}\frac{e^{x}-1}{x}=1$$

Examples

$$\lim_{x\to 0}\frac{1-\cos 2x}{x\sin x}=\lim_{x\to 0}\frac{1-\left(\cos^2 x-\sin^2 x\right)}{x\sin x}=\lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{x\sin x}=\lim_{x\to 0}\frac{2\sin^2 x}{x\sin x}=\underline{\underline{2}}$$

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\frac{3x\dfrac{\sin 3x}{3x}}{5x\dfrac{\sin 5x}{5x}}=\underline{\underline{\frac{3}{5}}}$$

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}=\lim_{x\to\infty}\left(\frac{x+1-1}{x+1}\right)^{x}=\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^{x}=$$

$$=/-\frac{1}{x+1}=t\Rightarrow \begin{matrix} x+1=-\dfrac{1}{t} \\ t\to 0\end{matrix}\ /=\lim_{t\to 0}(1+t)^{-\frac{1}{t}-1}=\lim_{t\to 0}\frac{1}{(1+t)(1+t)^{\frac{1}{t}}}=\underline{\underline{\frac{1}{e}}}$$

Evaluation of limits for expressions:

All basic operations (+, - ,* , /) have a simple position in the evaluation of limits:

(limit of an addition of two expressions is equal to the addition of these two limits,... etc.)

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=A+B$$

$$\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=A-B$$

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=A\cdot B$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{A}{B}\qquad B\neq 0$$

Left and right limits:

When we let "*x* approach *a*" we allow *x* to be both larger or smaller than *a*, as long as *x* gets close to *a*.

If we explicitly want to study the behaviour of *f(x)* as *x* approaches *a* through values larger (lower) than *a*, then we write

a right-limit (or limit from the right-hand side):

$$\lim_{x \searrow a} f(x) \text{ or } \boxed{\lim_{x \to a+} f(x)} \text{ or } \lim_{x \to a+0} f(x) \text{ or } \lim_{x \to a, x > a} f(x)$$

and a left-limit (or limit from the left-hand side):

$$\lim_{x \nearrow a} f(x) \text{ or } \boxed{\lim_{x \to a-} f(x)} \text{ or } \lim_{x \to a-0} f(x) \text{ or } \lim_{x \to a, x < a} f(x)$$

All four notations are in use (in various text-books).

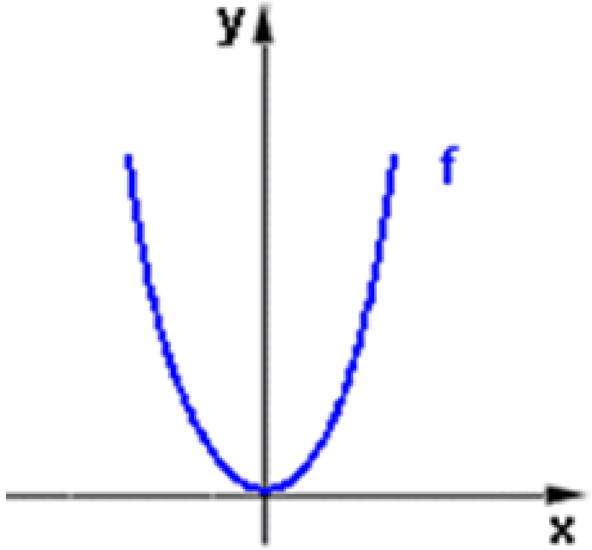
Relation of the limit of a function to continuity:

The notion of the limit of a function is very closely related to the concept of continuity.

Definition: A function *f(x)* is said to be continuous at *a* if it is both defined at *a* and its value at *a* equals the limit of f(x) as *x* approaches *a*:

$$\lim_{x \to a} f(x) = f(a)$$

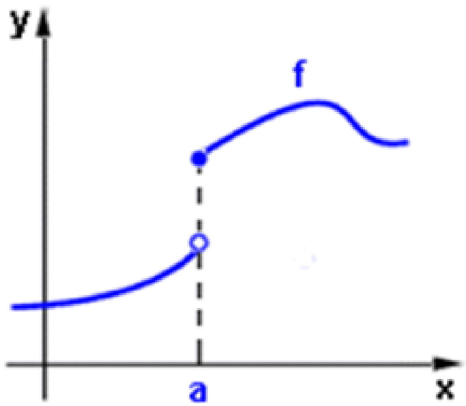
In other words: a continuous function is smooth, without any “steps”.



example of a continuous function.

Discontinuous function *f(x)* is a function, which for certain values or between certain values of the variable *x* does not vary continuously as the variable *x* increases or decreases.

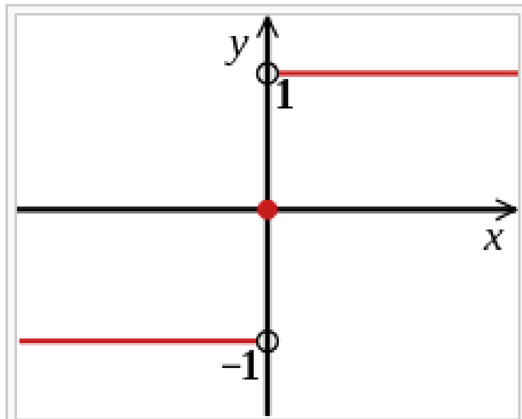
In other words: a discontinuous function can have “steps”.



example of a discontinuous function.

Example: the so called signum or sign function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Plot of the signum function. The hollow dots indicate that sgn(*x*) is 1 for all *x*>0 and −1 for all *x*<0.

Relation of the limit of a function to continuity:

For continuous functions it must be valid that the left-limit is equal to the right-limit (this is valid for the majority of cases):

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

For discontinuous functions this condition is invalid,

the left-limit is not equal to the right-limit:

$$\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$$

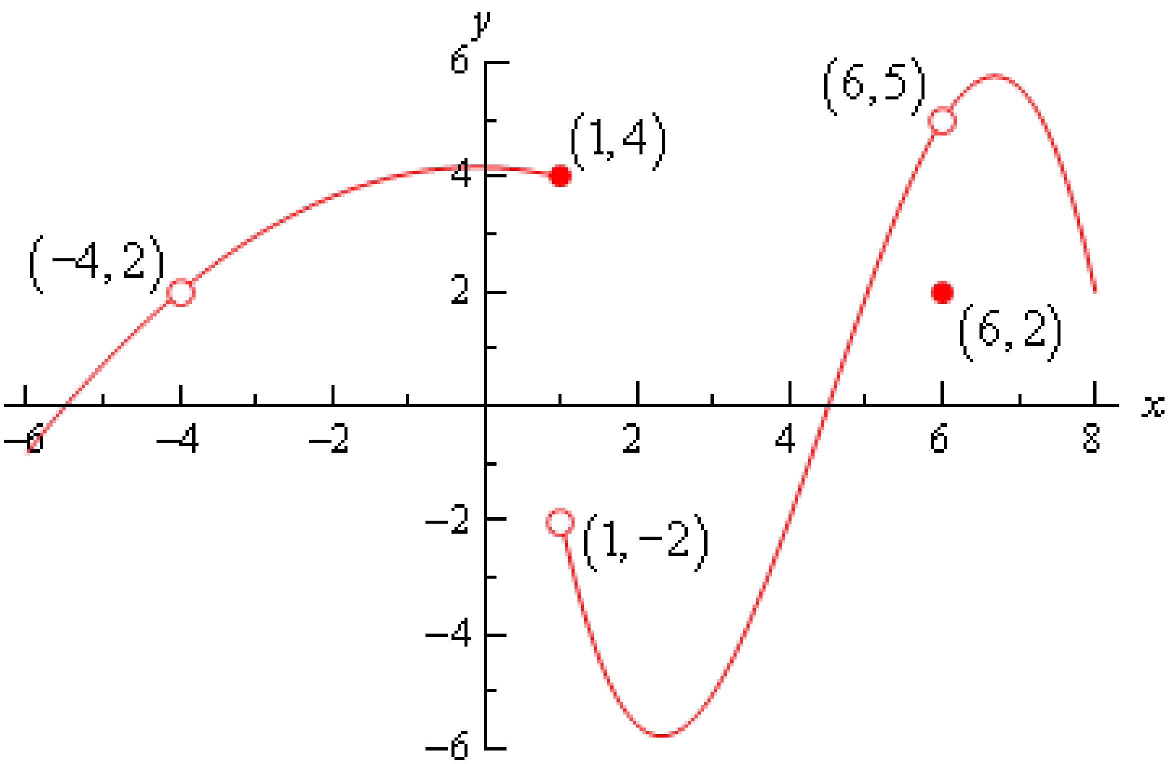
Example (discontinuous function):

$$\lim_{x \to 0+} \frac{|\sin x|}{\sin x} = 1, \quad \text{but} \quad \lim_{x \to 0-} \frac{|\sin x|}{\sin x} = -1.$$

Relation of the limit of a function to continuity:

For continuous functions it must be valid that the left-limit is equal to the right-limit (this is valid for the majority of cases):

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$



... but not for this one (x=6)