Mathematics for Biochemistry

LECTURE 6

Sequences, Series

Sequences of (real) numbers

Description: Sequence is an ordered collection of elements (numbers) from a set where repetitions are allowed.

Like a set, it contains members (also called elements, or terms).

The number of elements (possibly infinite) is called the length of the sequence.

The usual notation for a real number sequence is:

$$\{a_n\}_{n=1}^{M} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty} \quad \text{or} \quad a_1, a_2, \ldots$$

(n is a natural number, giving the sequence number of an element)

The total number of elements (length) can be a finite number (M) or Infinity (∞) – from this point of view we recognize finite and infinite sequence.

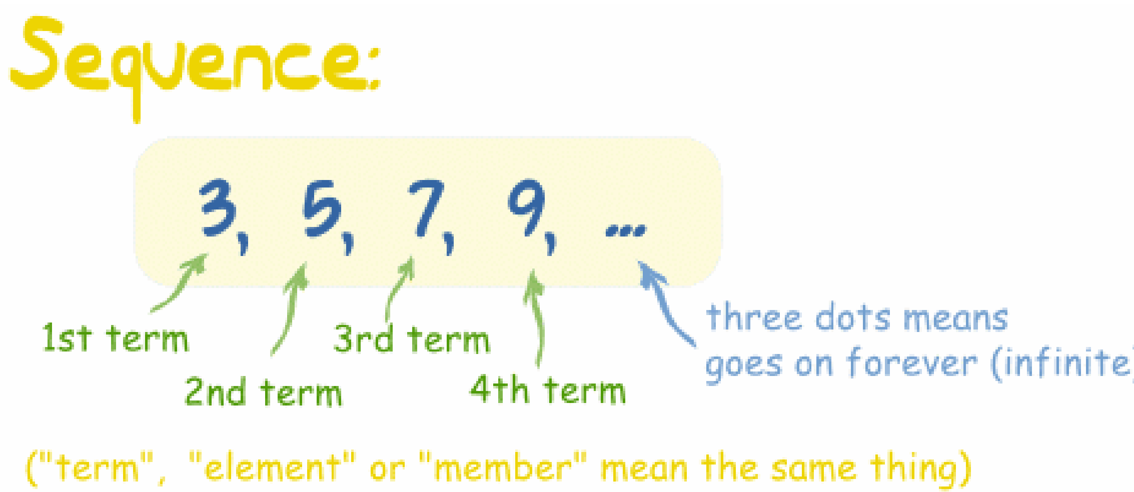
Sequences of (real) numbers

Formal definition:

A sequence is a function from a subset of the natural numbers to the real numbers set. In other words, a sequence is a map f(n) : N → R.
We can simply write - for all n is valid a_n : N → R.

Indexing: The terms of a sequence are commonly denoted by a single variable, say a_n, where the index n indicates the n^{th} element of the sequence.

$$\begin{array}{rcr}
a_1 & \leftrightarrow & \text{1st element} \\
a_2 & \leftrightarrow & \text{2nd element} \\
a_3 & \leftrightarrow & \text{3rd element} \\
\vdots & & \vdots \\
a_{n-1} & \leftrightarrow & \text{(n-1)th element} \\
a_n & \leftrightarrow & \text{nth element} \\
a_{n+1} & \leftrightarrow & \text{(n+1)th element} \\
\vdots & & \vdots
\end{array}$$



Sequences of (real) numbers

1. A sequence can be given by the list of its terms:

$a_1 = 1,\ a_2 = 2,\ a_3 = 4,\ a_4 = 7,\ a_5 = 11,\ \ldots$

2. A sequence may be also defined by giving an explicit formula for the n^{th} term.

E.g.:

$$a_n = \frac{1}{n},\quad n = 1, 2, \ldots$$

it is a sequence of terms: $a_1 = 1,\ a_2 = 1/2,\ a_3 = 1/3,\ \ldots$

3. A sequence may also be defined inductively - by recursion.

E.g.:

$$a_1 = 0,\ a_2 = 1,\ a_{n+2} = \frac{a_n + a_{n+1}}{2},\quad n = 1, 2, \ldots$$

it is a sequence of terms: $a_1 = 0,\ a_2 = 1,\ a_3 = 1/2,\ a_4 = 3/4,\ \ldots$

Sequences of (real) numbers

How to find "the next number" in a sequence?

There exist several rules that work, but no one of them is a general rule and very often a „trial-and-error method“ must be used.

One interesting method - based on finding differences (or divisions) between each pair of terms.

Example:

What is the next number in the sequence 7, 9, 11, 13, 15, ... ?

Solution: The differences are always 2, so, we can guess that "2n" is part of the answer. Let us try 2n. Such a model is wrong by 5, So, the right answer is:

$$\boxed{a_n = 2n + 5.}$$

n:	1	2	3	4	5
Terms :	7	9	11	13	15
2n:	2	4	6	8	10
Wrong by:	5	5	5	5	5

Some very important kind of sequences:

1. Arithmetic sequence:

In an Arithmetic Sequence the difference between one term and the next is a constant. In other words, we just add the same value each time ... infinitely.

Example: 1, 4, 7, 10, 13, 16, 19, 22, 25, ...

This sequence has a difference of 3 between each number.

In general, we can write an arithmetic sequence in a form:

$$\{a, a+d, a+2d, a+3d, \cdots\}$$

where:

a is the first term, and

d is the difference between the terms (called the "common difference")

We can write an arithmetic sequence as a rule:

$$\boxed{a_n = a + d(n-1)}$$

(we use here "n–1" because d is not used in the 1^{st} term).

Some very important kind of sequences:

2. Geometric sequence:

In a Geometric Sequence each term is found by multiplying the previous term by a constant.

Example: 2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number. Each term (except the first term) is found by multiplying the previous term by 2.

In general, we write a geometric sequence in a form:

$$\left\{a, aq, aq^2, aq^3, \cdots\right\}_{n=1}^{M}$$

where:

a is the first term, and

q is the factor between the terms (called common ratio or quotient)

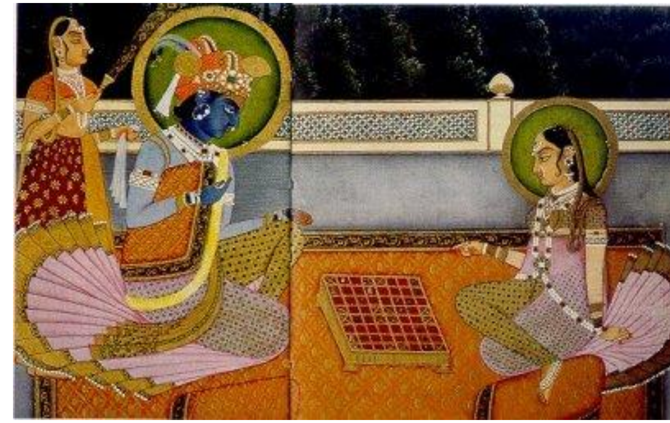
(q can not be equal zero - we would get a sequence {a,0,0,...})

Also here, we can write a geometric sequence as a rule:

$$\boxed{a_n = aq^{n-1}}$$

(we use "n-1" because aq^0 is for the 1^{st} term).

2. Geometric sequence:

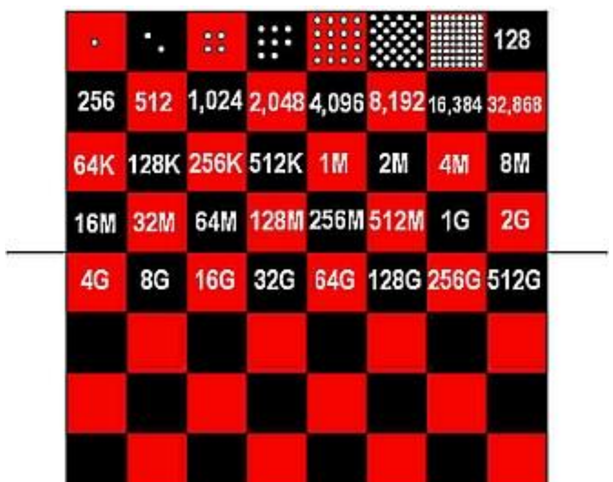


Legend of Paal Paysam (1/2)

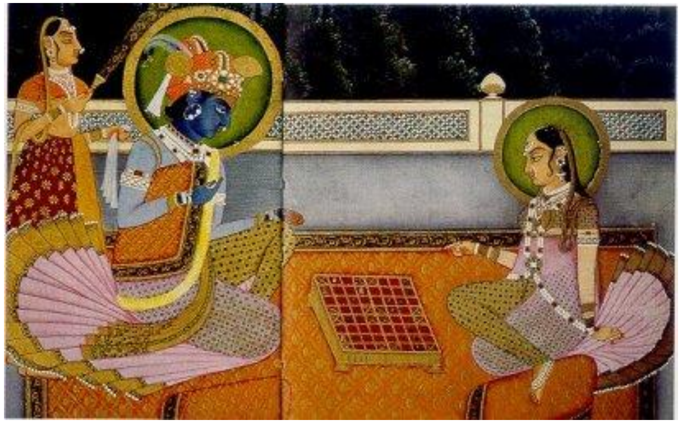
The legend goes that the tradition of serving Paal Paysam to visiting pilgrims started after a game of chess between the local king and the lord Krishna himself:

The king was a big chess enthusiast and had the habit of challenging wise visitors to a game of chess. One day a traveling sage was challenged by the king. To motivate his opponent the king offered any reward that the sage could name. The sage modestly asked just for a few grains of rice in the following manner: the king was to put a single grain of rice on the first chess square and double it on every consequent one.

Having lost the game and being a man of his word, the king ordered a bag of rice to be brought to the chess board. Then he started placing rice grains according to the arrangement:
1 grain on the first square, 2 on the second, 4 on the third, 8 on the fourth and so on:



2. Geometric sequence:

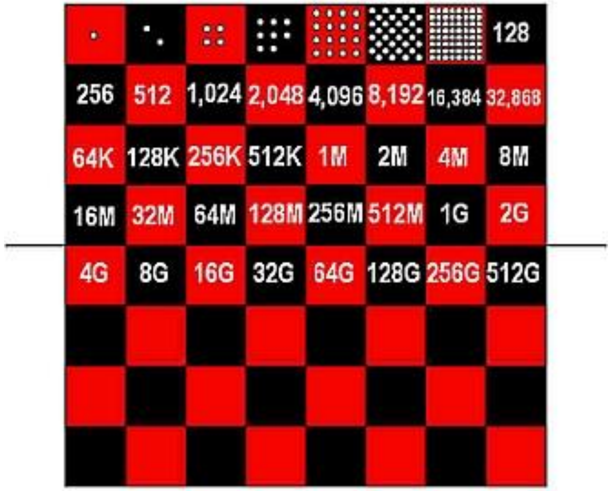


Legend of Paal Paysam (2/2)

The legend goes that the tradition of serving Paal Paysam to visiting pilgrims started after a game of chess between the local king and the lord Krishna himself:

Following the exponential growth of the rice payment the king quickly realized that he was unable to fulfill his promise because on the twentieth square the king would have had to put 1,000,000 grains of rice. On the fortieth square the king would have had to put 1,000,000,000 grains of rice. And, finally on the sixty fourth square the king would have had to put more than 9,000,000,000,000,000,000 grains of rice which (summed up) is equal to about 210 billion tons and is allegedly sufficient to cover the whole territory of India with a meter thick layer of rice.

$$18,446,744,073,709,551,615$$

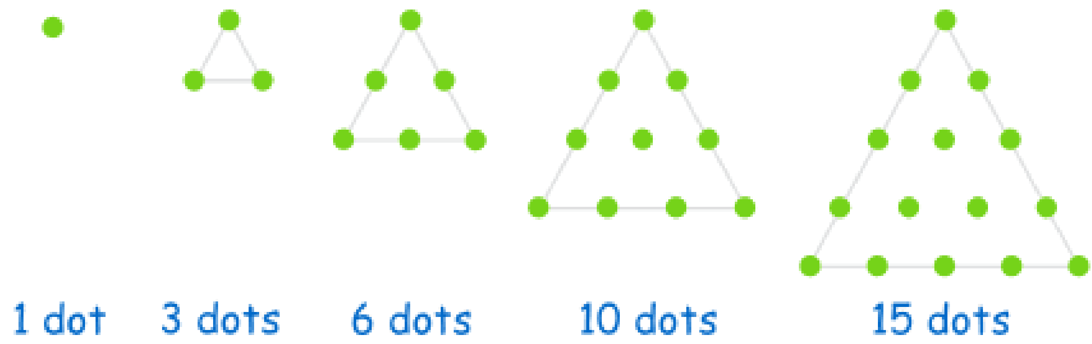


Some interesting kind of sequences:

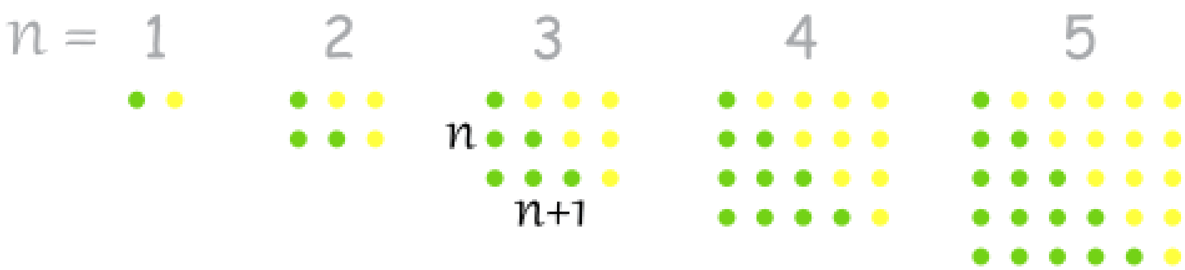
3. So called triangular number sequence :

1, 3, 6, 10, 15, 21, 28, 36, 45, ...

This sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence:



Finding the rule - rearranging the dots and form them into a rectangle:

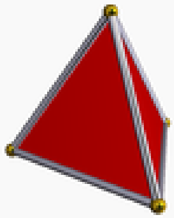
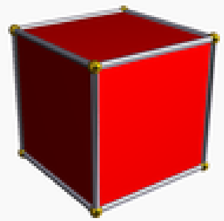
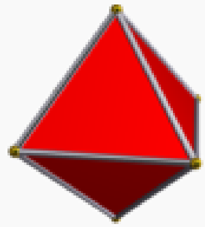
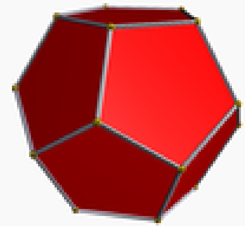
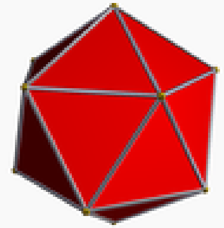


we get finally: $a_n = n(n+1)/2$

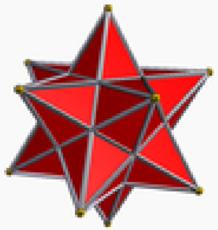
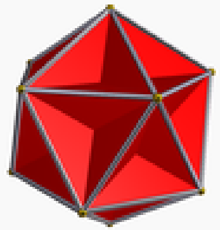
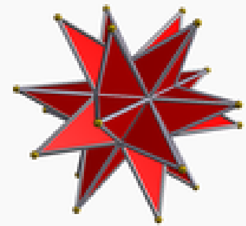
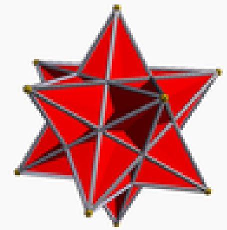
triangular number sequence :

and where we can use this?

Platonic solids

				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}
$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$	$\chi = 2$

Kepler-Poinsot polyhedra

			
Small stellated dodecahedron {5/2, 5}	Great dodecahedron {5, 5/2}	Great stellated dodecahedron {5/2, 3}	Great icosahedron {3, 5/2}
$\chi = -6$	$\chi = -6$	$\chi = 2$	$\chi = 2$

shapes of basic regular polyhedral bodies

Properties of sequences:

Increasing and decreasing:

A sequence is said to be monotonically increasing if each consecutive term is greater than or equal to the one before it.

If each consecutive term is strictly greater than (>) the previous term then the sequence is called strictly monotonically increasing.

A sequence is monotonically decreasing if each consecutive term is less than or equal to the previous one.

If each consecutive term is strictly smaller than (<) the previous term then the sequence is called strictly monotonically decreasing.

If a sequence is either increasing or decreasing, it is called a monotone sequence.

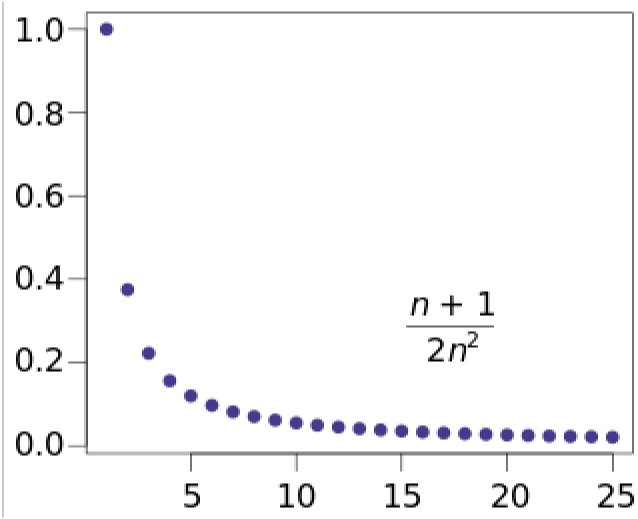
Properties of sequences:

Limits and convergence:

One of the most important properties of a sequence is convergence. Informally, a sequence converges if it has a limit – a sequence has a limit if it approaches some value *L*, called the limit, as *n* becomes very large:

$$\lim_{n \to \infty} a_n = L$$

example:



More precisely - the sequence converges if there exists a limit *L* such that the remaining a_n terms are arbitrarily close to *L* for some *n* large enough.

Properties of sequences:

Limits and convergence:

If a sequence converges to some limit, then it is convergent; otherwise, it is divergent.

If a_n gets arbitrarily large as $n \to \infty$ we write

$$\lim_{n\to\infty} a_n = \infty.$$

In this case we say that the sequence (a_n) *diverges*, or that it converges to infinity.

If a_n becomes arbitrarily "small" negative numbers (large in magnitude) as $n \to \infty$ we write

$$\lim_{n\to\infty} a_n = -\infty$$

and say that the sequence diverges or converges to minus infinity.

Series

Description: Informally speaking, series is the sum of the terms of a sequence:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_N &= a_1 + a_2 + a_3 + \cdots \\ &\vdots \end{aligned}$$

Partial sum: $$S_N = \sum_{n=1} a_n$$

If the sequence is infinite, the limit of the partial sum is named series:

$$\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n$$

If this limit exists (does not exist), the series is convergent (divergent).

Series

1. Arithmetic series:

Summing an arithmetic series:

To sum up the terms of arithmetic sequence:

$$a + (a + d) + (a + 2d) + \cdots$$

use this formula:

$$\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}\left[2a + (n-1)d\right]$$

where ***d*** is the common difference and ***n*** the length (number of terms) of the sequence.

1. Summing an arithmetic sequence - example:

Example: Add up the first 10 terms of the arithmetic sequence:

{ 1, 4, 7, 10, 13, ... }

The values of **a**, **d** and **n** are:

- **a** = **1** (the first term)
- **d** = **3** (the "common difference" between terms)
- **n** = **10** (how many terms to add up)

So:

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n}{2}(2a+(n-1)d)$$

Becomes:

$$\sum_{k=0}^{10-1}(1+k\cdot 3) = \frac{10}{2}(2\cdot 1+(10-1)\cdot 3)$$

= 5(2+9·3) = 5(29) = 145

2. Geometric series:

Summing a geometric series:

To sum up the terms of geometric sequence:

$$a + aq + aq^2 + \cdots aq^{n-1}$$

use this formula (valid for $q \neq 1$):

$$\sum_{k=0}^{n-1} aq^k = a\left(\frac{1-q^n}{1-q}\right)$$

where ***a*** is the first term, ***q*** the common ratio or quotient and ***n*** the length of the sequence.

Important: This formula can be simplified for convergent infinite series, when the parameter ***q*** fulfils the following condition: $|q| < 1$:

$$\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q}$$

2. Summing a geometric sequence - example:

Example: Sum the first 4 terms of

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number.

The values of **a**, **r** and **n** are:

- **a = 10** (the first term)
- **r = 3** (the "common ratio")
- **n = 4** (we want to sum the first 4 terms)

So:

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

Becomes:

$$\sum_{k=0}^{4-1} (10 \cdot 3^k) = 10\left(\frac{1-3^4}{1-3}\right) = 400$$

You can check it yourself:

10 + 30 + 90 + 270 = 400

Infinite series

$$s=\sum_{n=1}^{\infty} a_n \quad \to \begin{cases} <\infty \to \text{convergent series} \\ =\infty \to \text{divergent series} \end{cases}$$

Necessary (but not sufficient) condition for convergence: $\lim\limits_{n\to\infty} a_n = 0$

Some convergence criteria

1. d'Alembert's ratio test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \quad \to \begin{array}{l} L<1,\ \text{convergent series} \\ L>1,\ \text{divergent series} \\ L=1, \text{test fails, need of another test} \end{array}$$

2. Cauchy's limit test

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|} \quad \to \begin{array}{l} L<1,\ \text{convergent series} \\ L>1,\ \text{divergent series} \\ L=1, \text{test fails, need of another test} \end{array}$$

3. Raab's limit test

$$L=\lim_{n\to\infty}\left[n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)\right] \quad \to \begin{array}{l} L<1,\ \text{convergent series} \\ L>1,\ \text{divergent series} \\ L=1, \text{test fails, need of another test} \end{array}$$

Example 1: Find the nth member of the sequence: $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\cdots$

Solution 1: $a_n=\underline{\underline{\frac{1}{3n-2}}}$

Example 2: Find the nth member of the sequence: $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+\cdots$

Solution 2:

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\cdots=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots\Rightarrow a_n=\frac{1}{2n-1}\cdot\frac{1}{2n+1}=\underline{\underline{\frac{1}{4n^2-1}}}$$

Example 3: Find the summation of infinite sequence: $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\cdot\frac{1}{n+2}\right)$

Solution 3: $a_n=\frac{1}{n+1}\cdot\frac{1}{n+2}=\frac{1}{n+1}-\frac{1}{n+2}$

Partial sum: $s_n=\sum_{n=1}^{N}a_n=\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=1}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{n=2}+\underbrace{\left(\frac{1}{4}-\frac{1}{5}\right)}_{n=3}+\cdots+\frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{2}-\frac{1}{n+2}$

$$s=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\underline{\underline{\frac{1}{2}}}$$

Example 4: Find the summation of infinite sequence : $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{3}\right)^n$

Solution 4:

$$s_n=\sum_{n=1}^{N}a_n=\underbrace{\left(\frac{2}{3}\right)}_{n=1}-\underbrace{\left(\frac{2}{3}\right)^2}_{n=2}+\underbrace{\left(\frac{2}{3}\right)^3}_{n=3}-\underbrace{\left(\frac{2}{3}\right)^4}_{n=4}+\underbrace{\left(\frac{2}{3}\right)^5}_{n=5}-\cdots+(-1)^{n-1}\left(\frac{2}{3}\right)^n=$$

$$=\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^5+\cdots+\left(\frac{2}{3}\right)^{2n-1}-\left[\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^6+\cdots+\left(\frac{2}{3}\right)^{2n}\right]\Rightarrow$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{3}\right)^n=\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{2n-1}-\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{2n}=\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{2n-1}-\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{2n-1}=\left(1-\frac{2}{3}\right)\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{2n-1}=$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{2n-1}=\frac{1}{3}\left[\underbrace{\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^5+\cdots+\left(\frac{2}{3}\right)^{2n-1}}_{\text{geometrical sequence: } a_1=\frac{2}{3},\ q=\left(\frac{2}{3}\right)^2}\right]=\frac{1}{3}\left[\frac{a_1}{1-q}\right]=\frac{1}{3}\left[\frac{\frac{2}{3}}{1-\left(\frac{2}{3}\right)^2}\right]=\underline{\underline{\frac{2}{5}}}$$

Example 5: Check the series convergence: $$\sum_{n=1}^{\infty} \frac{2n}{7n+1}$$

Solution 5: Series is divergent $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{7n+1} = \frac{2}{7} > 0$$

Example 6: Check the series convergence: $$\sum_{n=1}^{\infty} \left(\frac{n+2}{2n-1}\right)^n$$

Solution 6: Series is convergent (Cauchy's limit test)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+2}{2n-1}\right)^n} = \lim_{n\to\infty} \frac{n+2}{2n-1} = \frac{1}{2} < 1$$

Generalized harmonic series

$$1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha}+\frac{1}{5^\alpha}+\frac{1}{6^\alpha}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^\alpha}, \qquad \alpha>0$$

- convergent for $\alpha>1$
- divergent for $\alpha<1$

harmonic series $$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\to\infty$$

alternate harmonic series $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\ln 2$$

squares series $$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{25}+\cdots=\frac{\pi^2}{6}$$