Mathematics for Biochemistry

LECTURE 7

Differentiation 1

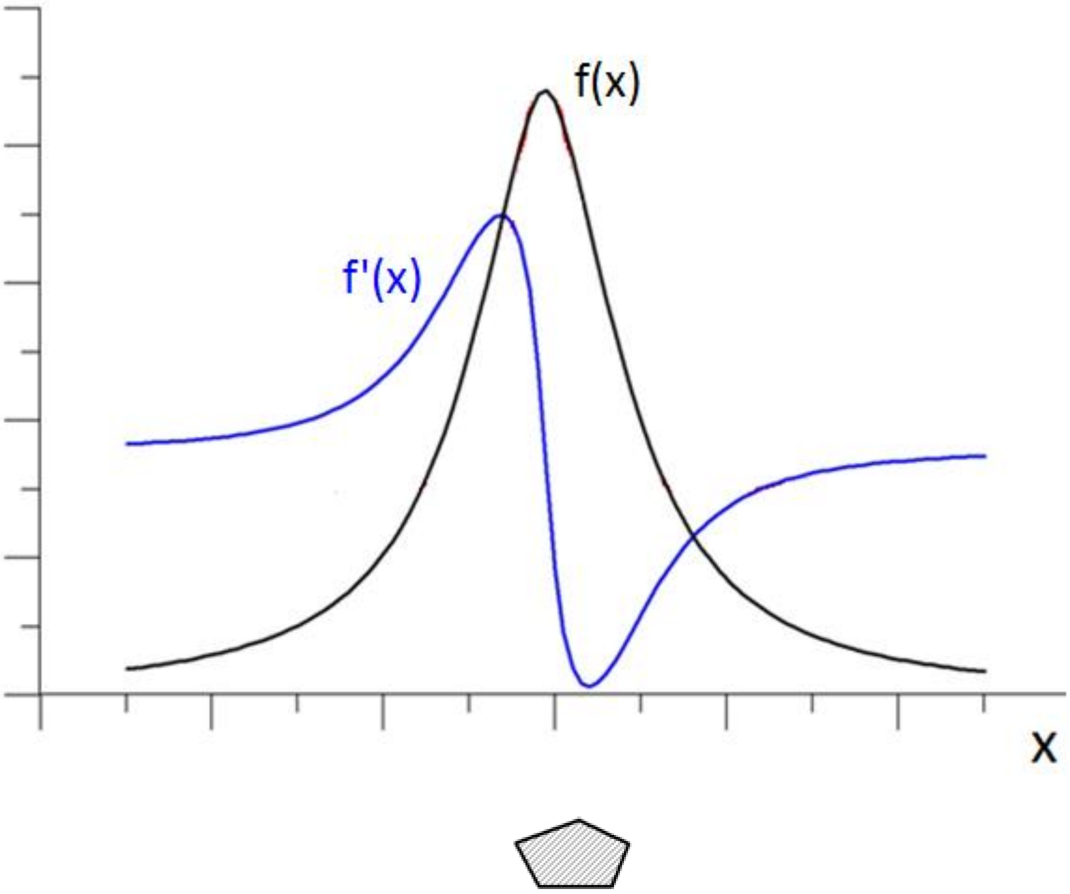
Lecture 7: Differentiation

Content:

- introduction
- basic examples (using the definition)
- non-differentiable functions
- basic differentiation rules
- derivatives of elementary functions

Differentiation - introduction

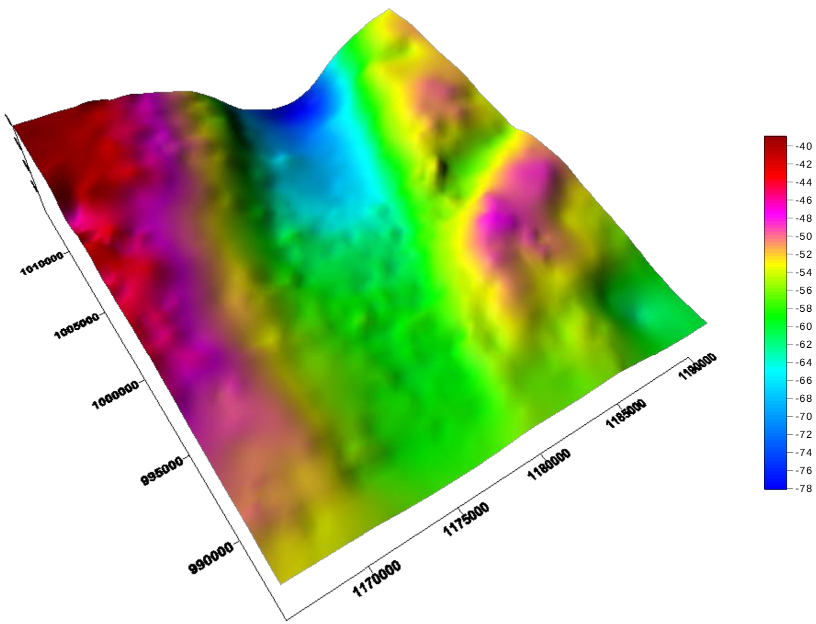
- Differentiation (or **derivative**) is, together with integrals, one of the most important tools in mathematical calculus.
- Derivative measures the sensitivity of the function f(x) to the change of the independent variable (x) - we can say that the derivative of a function f(x) of a variable x is a measure of the rate at which the value of the function changes with respect to the change of the variable.
- In a very simple form, we can say that a derivative gives the change of function f(x) for a very small change of x.
- It is called the derivative of f with respect to x.
- Many applications in physics and other natural sciences are built upon the use of derivatives (e.g. derivative of the position of a moving object with respect to time is the object's velocity).



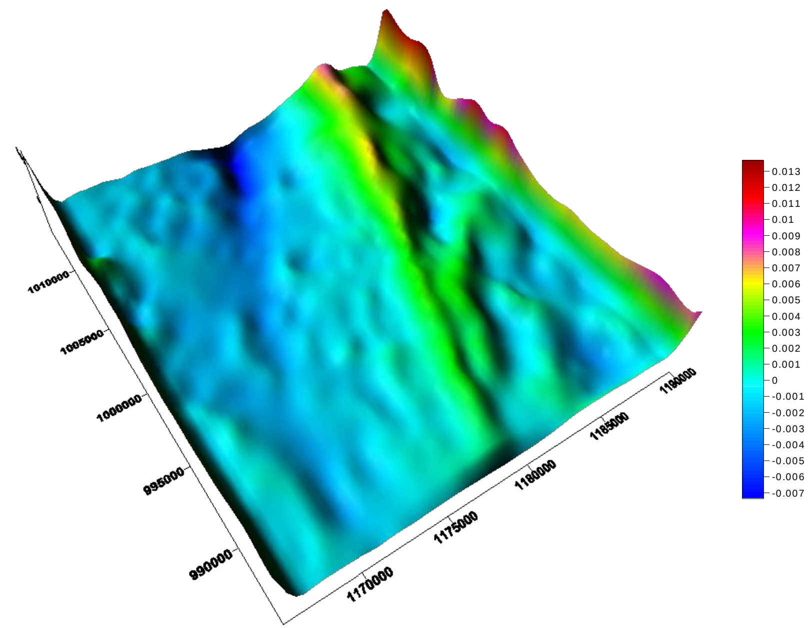
Example from physics:

Size of gravitational attraction of a mass f(x) (black curve) and its derivative f ’(x) (blue curve)

Comment: Both curves are plotted in one graph – this is incorrect (vertical scales are different).



anomalous gravity field (Dead Sea)



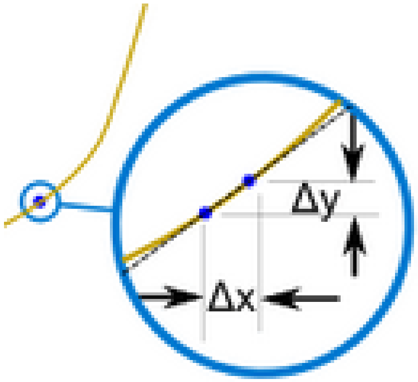
its derivative with respect to x

A next example from physics:

Map of anomalous gravitational attraction in Dead Sea (left) and its derivative with respect to x-coordinate (right).

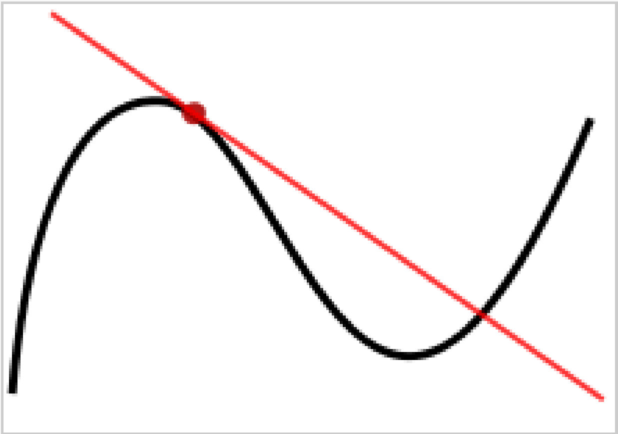
Derivatives - introduction

- derivative is often described as the "instantaneous rate of change“
- the process of finding a derivative is called differentiation. The reverse process is called antidifferentiation. The *fundamental theorem of calculus* states that antidifferentiation is the same as integration.



$$\text{slope } m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

- the derivative of a function of a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point; this means that it describes the best linear approximation of the function near that input value.



The graph of a function, drawn in black, and a tangent line to that function, drawn in red. The slope of the tangent line is equal to the derivative of the function at the marked point.

Derivatives - notation

Two distinct notations are commonly used for the derivative:

one coming from Gottfried Wilhelm Leibniz: $\frac{dy}{dx}$

and the second from Joseph Louis Lagrange: $y'(x)$

(sometimes signed also as Newton's notation - $\dot{y}$)

1. Leibniz's notation is suggesting the ratio of two infinitesimal quantities – this expression is read as: "the derivative of *y* with respect to *x*" or "*dy* by *dx*".
2. In Lagrange's notation the derivative with respect to *x* of a function *y*(*x*) is denoted *y*'(*x*) – this expression is read as: "*y* prime *x*" or "*y* dash *x*".

Comment: In mathematics, infinitesimals are things so small that there is no way to measure them.

Derivatives - notation

Sir Isaac Newton (left) and Gottfried Wilhelm von Leibniz (right)

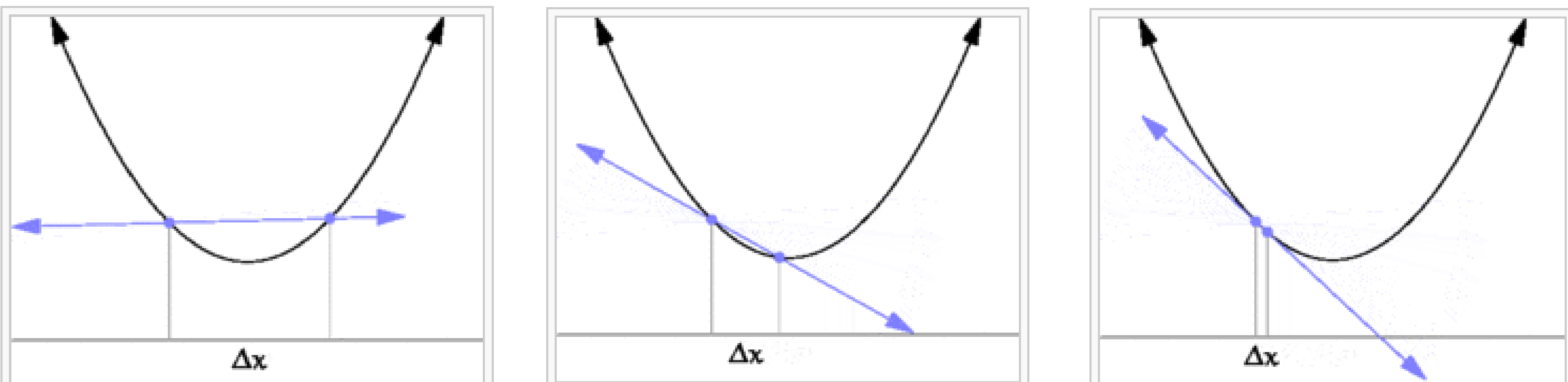
Leibniz–Newton calculus controversy

Derivatives – basic definition

Definition:

The derivative of $f(x)$ with respect to x is the function $f'(x)$:

$$\boxed{\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$



A secant approaches a tangent

A secant approaches a tangent

A secant approaches a tangent

when Δx is getting smaller then the ratio of $\Delta f/\Delta x$ is getting closer to the slope (derivative) of the function in this point

Derivatives – definition

Definition:

The derivative of *f*(*x*) with respect to *x* is the function *f* ’(*x*):

$$\boxed{\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

function f(x) is differentiable, when this limit exists and when it exists in every point of a defined interval.

Equivalent notations:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(x)_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

<u>Comment:</u>

Derivative of a function is again a function!

Derivatives – definition

- derivative is often described as the "instantaneous rate of change“
- in a very simple form we can say that a derivative gives the change of function f(*x*) for a very small change of *x*
- the derivative of a function of a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point
- and how is the relationship „differentiable vs. continuous“?

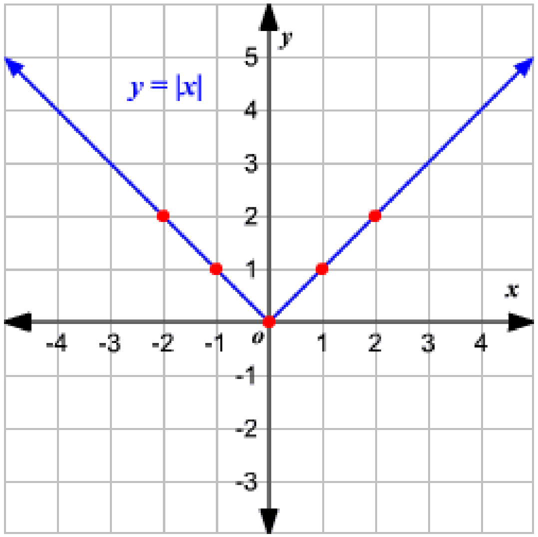
Differentiable implies Continuous

Theorem. *If a function f is differentiable at some a in its domain, then f is also continuous at a.*

But this statement is not a equivalency – if a function is continuous, it must not be also differentiable.

Derivatives – non-differentiable functions

Functions with „corner or sharp edge (cusp)“:



A graph with a corner. Consider the function

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

This function is continuous at all x, but it is not differentiable at $x = 0$.

To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0} \operatorname{sign}(x).$$

We know this limit does not exist.

If you look at the graph of $f(x) = |x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

Derivatives – non-differentiable functions

Functions with „corner or sharp edge (cusp)“:

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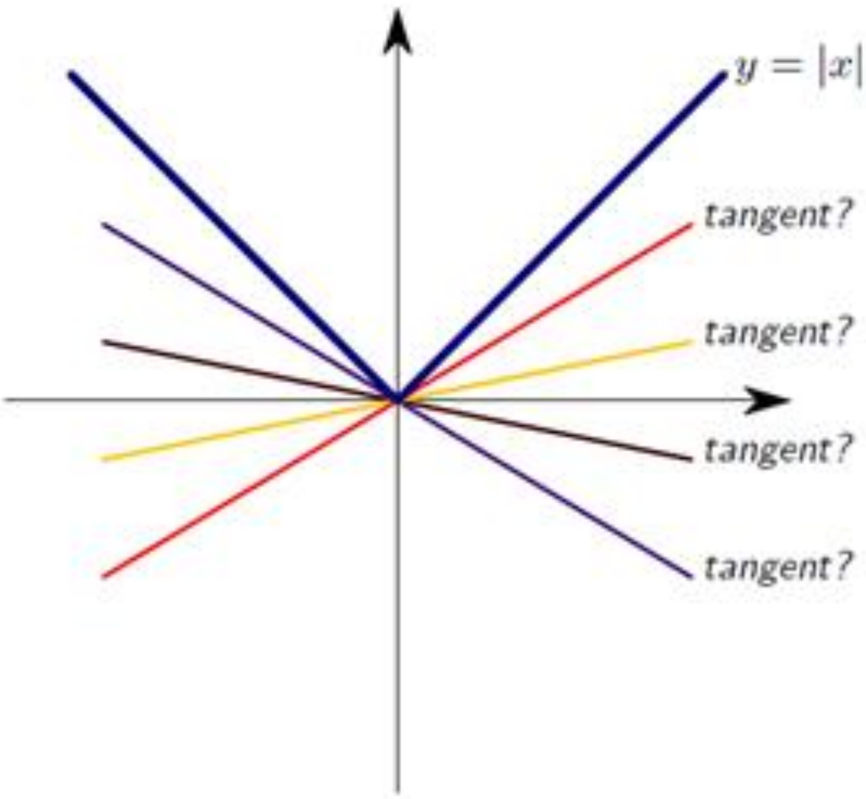

Figure The graph of $y = |x|$ has no tangent at the origin.

Derivatives – non-differentiable functions

Functions with „corner or sharp edge (cusp)“:

A graph with a cusp.

Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|x|}}{x} = ?$$

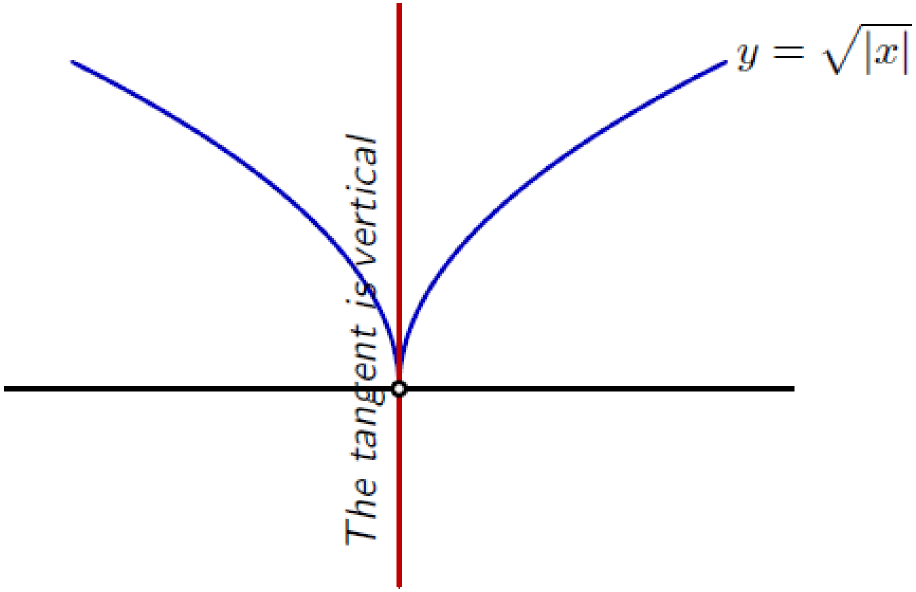

Figure Tangent to the graph of $y = |x|^{1/2}$ at the origin

Comment:

Somebody could argue the tangent is the vertical line (identical with the y-axis), but vertical lines do not have slopes (slope is infinity).

Derivatives – basic examples (using the definition)

Example 1

$$f(x) = x^2$$

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2x$$

Example 2

$$f(x) = x^3$$

$$f'(x) = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2)}{x - a} = 3x^2$$

Example 3

$$f(x) = c$$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - k(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

The derivative of any constant is zero

Derivatives – basic examples (using the definition)

We have obtained following sequence of solutions:

$(x^0)' = 0,$

$(x^1)' = 1,$

$(x^2)' = 2x,$

$(x^3)' = 3x^2,$

... would it be possible to write here some kind of generalization?

Derivative of x^n, where $n = 1, 2, 3,...$ is equal to:

$$\boxed{\left(x^n\right)' = nx^{n-1}}$$

Derivatives – basic differentiation rules

As is was also in the case of limits evaluation, the use of the basic definition for the evaluation of derivatives is sometimes very cumbersome, so we use some basic rules for differentiation:

1. Constant rule: $c' = 0$

2. Sum rule: $(u \pm v)' = u' \pm v'$

3. Product rule: $(u \cdot v)' = u' \cdot v + u \cdot v'$

4. Constant product rule: $\left(c \cdot f(x)\right)' = c \cdot f'(x)$

5. Quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$

6. Chain rule: $\left(f(x) \circ g(x)\right)' = f\left(g(x)\right) = f'\left(g(x)\right) \cdot g'(x)$

where *c* – constant, *u(x), v(x), f(x), g(x)* are functions

Derivatives – basic differentiation rules

<u>Simple examples</u> (product and quotient rule):

Differentiate the following function:

$$f(x) = x\left(3 + x^2\right)$$

Result: $f'(x) = 3\left(1 + x^2\right)$

Differentiate the following function:

$$f(x) = \frac{1+x}{1-x}$$

Result: $f'(x) = \dfrac{2}{(1-x)^2}$

Derivatives – basic differentiation rules

7. Product rule with more then one factor

$$\left(u_1 \cdot u_2 \cdot u_3\right)' = u_1' \cdot u_2 \cdot u_3 + u_1 \cdot u_2' \cdot u_3 + u_1 \cdot u_2 \cdot u_3'$$

$$\left(u_1 \cdots u_n\right)' = u_1' \cdots u_n + u_1 \cdot u_2' \cdots u_n + u_1 \cdot u_2 \cdots u_n'$$

8. Power rule: $\left(\left[f(x)\right]^n\right)' = n\left[f(x)\right]^{n-1} \cdot f'(x)$

$$\left(\frac{1}{\left[f(x)\right]^n}\right)' = \left(\left[f(x)\right]^{-n}\right)' \to \text{power rule}$$

Derivatives – basic differentiation rules

<u>Simple examples</u> (power rule, chain rule):

Differentiate the following function:

$$f(x) = \frac{1}{x^4}$$

Result: $f'(x) = -4x^{-5} = -\frac{4}{x^5}$

Differentiate the following function:

$$f(x) = \frac{1}{(1-2x)^2}$$

Result: $f'(x) = -\frac{2}{(1-2x)^3}(-2x) = \frac{4x}{(1-2x)^3}$

Derivatives – basic differentiation rules

<u>Simple examples</u> (power rule, chain rule):

Find the derivative of function: $f(x) = \frac{x-a}{\sqrt{x^2+b}}$

Solution: $\left[\frac{x-a}{\sqrt{x^2+b}}\right]' = \frac{(x-a)'\sqrt{x^2+b} - (x-a)\left(\sqrt{x^2+b}\right)'}{\left(\sqrt{x^2+b}\right)^2} =$

$$= \frac{1\cdot\sqrt{x^2+b} - (x-a)\frac{1}{2}\left(x^2+b\right)^{-1/2} 2x}{\left(x^2+b\right)} =$$

$$= \frac{\sqrt{x^2+b} - x(x-a)\left(x^2+b\right)^{-1/2}}{x^2+b} =$$

$$= \frac{xa+b}{\left(x^2+b\right)^{\frac{3}{2}}}$$

Derivatives – elementary functions

$$(c)' = \frac{d}{dx}(c) = 0$$

$$\left(x^p\right)' = \frac{d}{dx}\left(x^p\right) = px^{p-1}$$

$$\left(a^x\right)' = \frac{d}{dx}\left(a^x\right) = a^x \ln a \rightarrow \left(e^x\right)' = e^x$$

$$\left(\log_a x\right)' = \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \cdot \ln a} \rightarrow \left(\ln x\right)' = \frac{1}{x}$$

$$(\sin x)' = \frac{d}{dx}(\sin x) = \cos x$$

$$(\cos x)' = \frac{d}{dx}(\cos x) = -\sin x$$

$$(\tan x)' = \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$

$$(\cot x)' = \left(\frac{1}{\cot x}\right)' = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\frac{1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$(\text{arccot}\, x)' = \frac{d}{dx}(\text{arccot}\, x) = \frac{1}{1+x^2}$$

$$(\sinh x)' = \frac{d}{dx}(\sinh x) = \cosh x$$

$$(\cosh x)' = \frac{d}{dx}(\cosh x) = \sinh x$$

$$(\tanh x)' = \frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}$$

$$(\coth x)' = \frac{d}{dx}(\coth x) = -\frac{1}{\sinh^2 x}$$

Derivatives – elementary functions

Simple examples:

Find the derivative of function: $f\left(x\right)=3\cos^2 x\cdot\arctan x$

Solution: $\left[3\cos^2 x\cdot\arctan x\right]'=3\left[\left(\cos^2 x\right)'\cdot\arctan x+\cos^2 x\cdot\left(\arctan x\right)'\right]=$

$$=3\left[-2\cos x\sin x\cdot\arctan x+\frac{\cos^2 x}{1+x^2}\right]$$

Find the derivative of function: $f\left(x\right)=3^x+e^{-2x}-\log_{\frac{1}{2}}2x$

Solution: $\left[3^x+e^{-2x}-\log_{\frac{1}{2}}2x\right]'=3^x\ln 3-2e^{-2x}-\dfrac{1}{2x\ln\frac{1}{2}}\cdot 2=$

$$=3^x\ln 3-2e^{-2x}+\frac{1}{2x\ln 2}$$