Mathematics for Biochemistry

LECTURE 8

Differentiation 2

Find Derivative of $y = x^x$

A calculus tutorial on how to find the first derivative of $y = x^x$ for $x > 0$.

Note that the function defined by $y = x^x$ is neither a power function of the form x^k nor an exponential function of the form b^x and the formulas of Differentiation of these functions cannot be used. We need to find another method to find the first derivative of the above function.

If $y = x^x$ and $x > 0$ then $\ln y = \ln (x^x)$

Use properties of logarithmic functions to expand the right side of the above equation as follows.

$\ln y = x \ln x$

We now differentiate both sides with respect to x, using chain rule on the left side and the product rule on the right.

$y'(1 / y) = \ln x + x(1 / x) = \ln x + 1$, where $y' = dy/dx$

Multiply both sides by y

$y' = (\ln x + 1)y$

Substitute y by x^x to obtain

$y' = (\ln x + 1)x^x$

Derivatives – logarithmic differentiation

Simple example:

Find the derivative of function: $f(x) = (\sin x)^{\cos x}$

Solution:

$$\left[(\sin x)^{\cos x}\right]' = \left[e^{\ln(\sin x)^{\cos x}}\right]' = \left[e^{\cos x \ln(\sin x)}\right]' = e^{\cos x \ln(\sin x)} \cdot \left(\cos x \ln(\sin x)\right)' =$$

$$= e^{\cos x \ln(\sin x)} \cdot \left(-\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}\right) =$$

$$= -(\sin x)^{\cos x}\left(\sin x \ln(\sin x) + \frac{\cos x}{\tan x}\right)$$

Higher derivatives

Derivative of a function is again a function – so it can be differentiated again (and again). So, we can obtain a 2nd derivative of *f*(*x*), third or *n*th.

Notation:

$$f^{(0)}(x) = f^{(0)}(x); \quad f^{(1)}(x) = f'(x); \quad f^{(2)}(x) = f''(x); \quad \cdots$$

or:

$$\frac{df(x)}{dx} = f'(x); \quad \frac{d^2 f(x)}{dx^2} = f''(x); \quad \cdots; \quad \frac{d^n f(x)}{dx^n} = f^{(n)}(x)$$

Simple example:

$$f(x) = e^{3x} \rightarrow f'(x) = 3e^{3x}; \quad f''(x) = 9e^{3x}; \quad f'''(x) = 27e^{3x}$$

Derivatives utilization – Taylor series

Definition: The Taylor series of a real or complex-valued function *f*(x) that is infinitely differentiable at a real or complex number *a* is the power series:

$$f(x) \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

In other words: a function *f*(x) can be approximated in the close vicinity of the point *a* by means of a power series, where the coefficients are evaluated by means of higher derivatives evaluation (divided by corresponding factorials).

Taylor series can be also written in the more compact sigma notation (summation sign) as:

$$f(x) \doteq \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Comment: factorial 0! = 1.

Derivatives utilization – Taylor series

When *a* = 0, the Taylor series is called a MacLaurin series

(Taylor series are often given in this form):

$$f(x) \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

↓ *a* = 0

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

... and in the more compact sigma notation (summation sign):

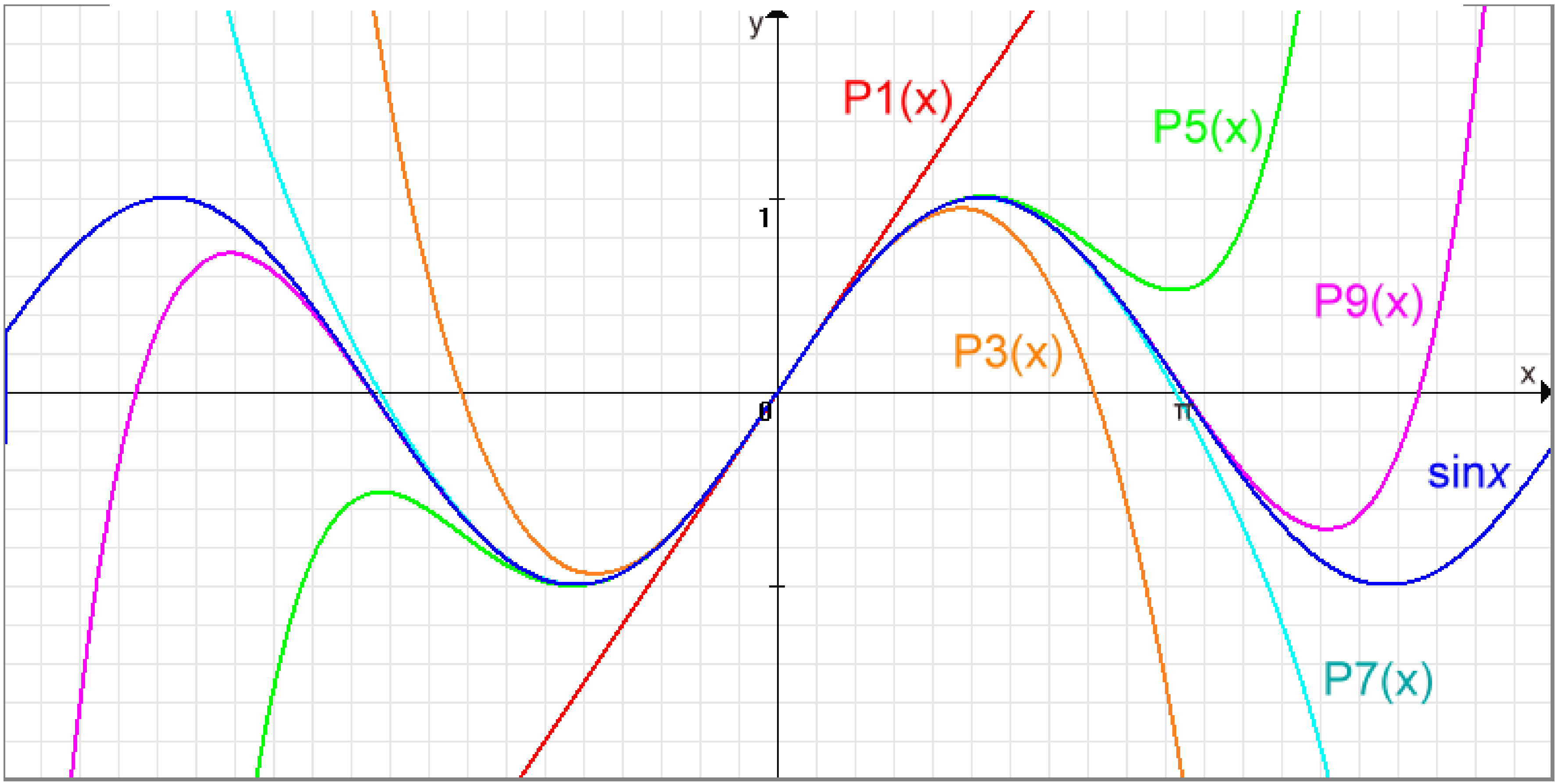
$$f(x) \doteq \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

↓ *a* = 0

$$f(0) \doteq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

graphical demonstration – approximation with polynomials (sin*x*)

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$



nice examples:

https://en.wikipedia.org/wiki/Taylor_series

http://mathdemos.org/mathdemos/TaylorPolynomials/

Derivatives utilization – Taylor (Maclaurin) series

Examples (Maclaurin series):

$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Derivatives utilization – Taylor (Maclaurin) series

Taylor (Maclaurin) series are utilized in various areas:

1. Approximation of functions – mostly in so called local approx. (in the close vicinity of point *a* or 0).
   Precision of this approximation can be estimated by so called remainder or residual term R_n, describing the contribution of ignored higher order terms:

$$f(x) \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \boxed{R_n(x)}$$

2. Simplification of solutions – so called linearization: in some solutions (|x| < 1) the higher degree terms can be ignored and only the constant and linear term are used (we do not have to work later on with more complicated higher degree polynomials).

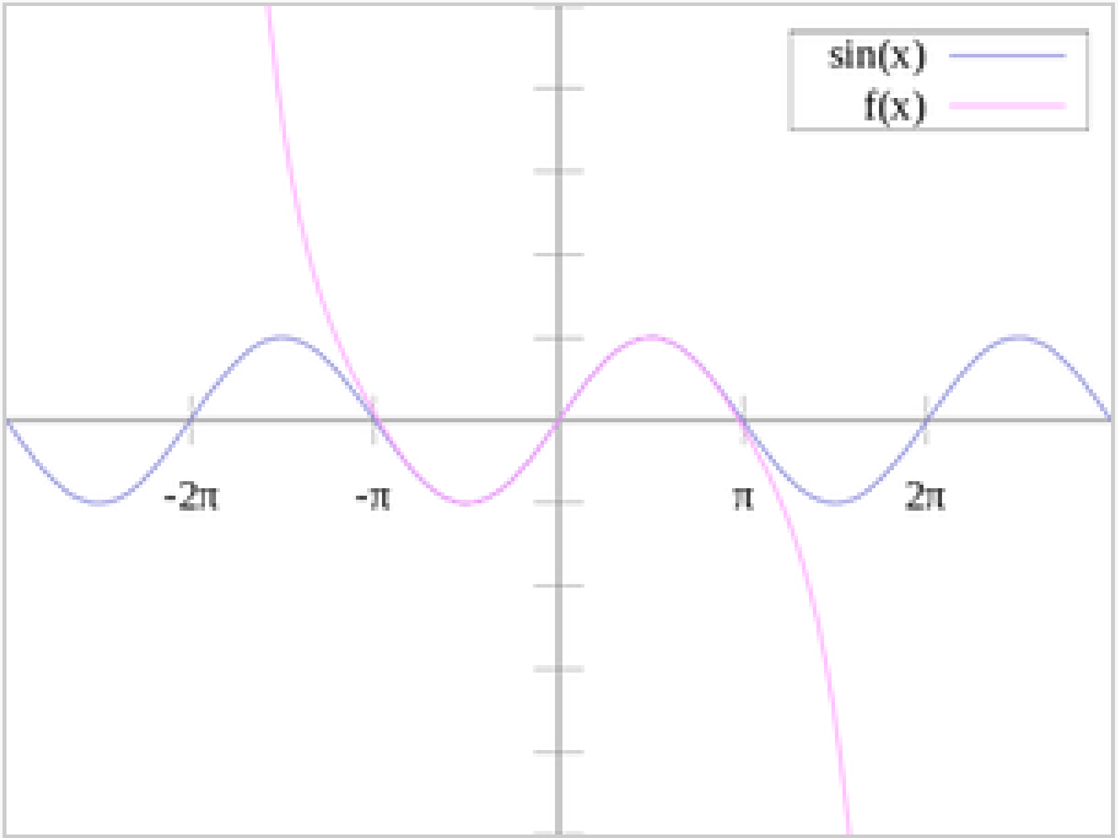
$$f(x) \doteq f(0) + \frac{f'(0)}{1!}x + \cancel{\frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3} + \cdots.$$

Derivatives utilization – Taylor (Maclaurin) series

Local approximation of functions (in local vicinity of an isolated point, not on whole interval):

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

The error in this approximation (for −1 < x < 1) is less than 0.000003.



The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full period centered at the origin.

L'Hospital's rule:

The evaluation of limits of indeterminate expressions (expressions of type 0/0, ∞/∞, 0^0, ∞^0 or 1^∞).

Description: L'Hospital's rule states that for functions *f* and *g* (which are differentiable on an interval *I*) it is valid:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

when $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ is valid for all x in I with $x \neq c$.

L'Hospital's rule:

Examples:

1. $f(x)=\dfrac{5x-2}{7x+3}\quad ; \quad x\to\infty$

Solution

$$\lim_{x\to\infty}\frac{5x-2}{7x+3}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(5x-2)'}{(7x+3)'}=\lim_{x\to\infty}\frac{5}{7}=\frac{5}{7}$$

---

2. $f(x)=\dfrac{\ln x}{\sqrt{x}}\quad ; \quad x\to\infty$

Solution

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(\ln x)'}{(\sqrt{x})'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{x}=0$$

Examples:

3. $f(x) = \sin x \ln x \quad ; \quad x \to 0^+$

Solution $\lim_{x\to 0^+} (\sin x \ln x) = [0\cdot(-\infty)] = \lim_{x\to 0^+} \frac{(\ln x)'}{\left(\frac{1}{\sin x}\right)'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} =$

$$= \lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x} = 0$$

---

4. $f(x) = \frac{1}{\ln x} - \frac{1}{x-1} \quad ; \quad x \to 1$

Solution

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = [\infty - \infty] = \lim_{x\to 1}\left(\frac{(x-1)-\ln x}{(x-1)\ln x}\right) = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x + \frac{(x-1)}{x}} =$$

$$= \lim_{x\to 1}\frac{x-1}{x\ln x + (x-1)} = \lim_{x\to 1}\frac{1}{\ln x + 1 + 1} = \frac{1}{2}$$

Examples:

5. $f\left(x\right)=\dfrac{\sin x}{x}\quad ;\quad x\to 0$

Solution

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\left(\sin x\right)'}{\left(x\right)'}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

This approach (L'Hospital rule) is **not correct** here:

$$\left(\sin x\right)'=\lim_{h\to 0}\frac{\sin\left(x+h\right)-\sin x}{h}=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}=$$

$$=\sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\boxed{\lim_{h\to 0}\frac{\sin h}{h}}=\cos x$$

To find out, "how much" is $\dfrac{\sin x}{x}\quad ;\quad x\to 0$

We need to know, "how much" is $\dfrac{\sin h}{h}\quad ;\quad h\to 0$

Circular reasoning = Logical fallacy

Examples:

6. $f(x) = \dfrac{x - \sin x}{e^x - e^{-x} - 2x} \quad ; \quad x \to 0$

Solution

$$\lim_{x\to 0} \frac{x - \sin x}{e^x - e^{-x} - 2x} = \lim_{x\to 0} \frac{1 - \cos x}{e^x + e^{-x} - 2} = \lim_{x\to 0} \frac{\sin x}{e^x - e^{-x}} =$$

$$= \lim_{x\to 0} \frac{e^x \sin x}{e^{2x} - 1} = \lim_{x\to 0} \frac{e^x \sin x + e^x \cos x}{2e^{2x}} = \lim_{x\to 0} \frac{\sin x + \cos x}{2e^x} = \frac{1}{2}$$

---

7. $f(x) = x^{\frac{1}{x}} \quad ; \quad x \to \infty$

Solution

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \left[\infty^0 \to x^{\frac{1}{x}} = t \Rightarrow \ln t = \ln x^{\frac{1}{x}} = \frac{\ln x}{x} \right] = \lim_{x\to\infty} \frac{\ln x}{x} = \left[\frac{\infty}{\infty} \right] = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\to \ln t \to 0 \Rightarrow t \to 1 \Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = 1$$