Mathematics for Biochemistry

LECTURE 9

Function analysis

Content:

- functions graph course analysis

Functions graph course analysis

Derivatives are an important tool in the analysis of function's properties – monotonicity, curvature, extremal values, asymptotes, range,… .

$$f(x) = \frac{2x^3}{x^2 - 1}$$

1. Domain of definition, roots, points of discontinuity
2. First differentiation, stationary points
3. Monotonicity
4. Second differentiation
5. Curvature, inflection points
6. Local extrema
7. Asymptotes
8. Graph
9. Range, other properties (sup, inf, even, odd, periodic)

$$f(x) = \frac{2x^3}{x^2 - 1}$$

1. Domain of definition, roots, points of discontinuity

$$x^2 - 1 \neq 0 \quad \Rightarrow \quad D(f) = \mathbb{R} - \{-1, 1\}$$

$$f(x) = 0 \rightarrow 2x^3 = 0 \quad \Rightarrow \quad \text{root:} \quad x = 0$$

$$\text{PoD} \rightarrow \{-1, 1\} \qquad \lim_{x \to BN_1^-} f(x) = -\infty; \lim_{x \to BN_1^+} f(x) = \infty; \lim_{x \to BN_2^-} f(x) = -\infty; \lim_{x \to BN_2^+} f(x) = \infty$$

2. First differentiation, stationary points

$$f(x) = \frac{2x^3}{x^2 - 1} \quad \rightarrow \quad f'(x) = \left(\frac{2x^3}{x^2 - 1}\right)' = \frac{6x^2(x^2 - 1) - 2x^3(2x)}{(x^2 - 1)^2} = \frac{2x^2(x^2 - 3)}{(x^2 - 1)^2}$$

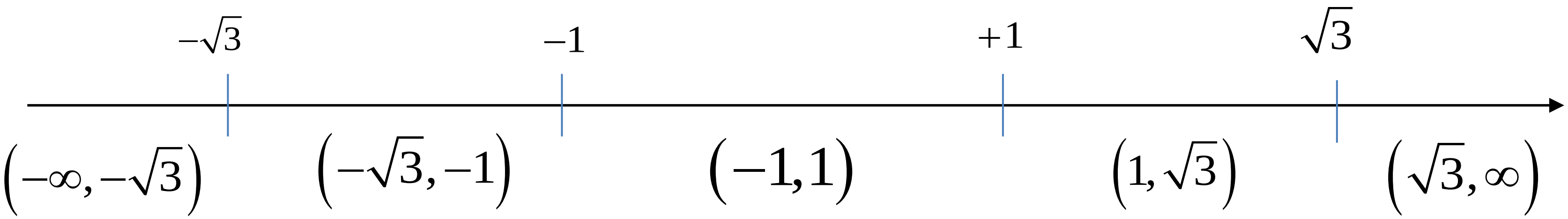
Stationary point (SP) – “extremist suspect” ☺

$$\text{SP} \rightarrow f'(x) = 0 \Rightarrow \frac{2x^2(x^2 - 3)}{(x^2 - 1)^2} = 0 \Rightarrow \left[\begin{array}{c} SP_1 = 0 \\ SP_2 = -\sqrt{3} \\ SP_1 = \sqrt{3} \end{array}\right.$$

3. Monotonicity

$$f(x)=\frac{2x^3}{x^2-1}$$

PoDs and SPs “divide” the real axis into intervals, where the monotonicity is studied with help of the first differentiation’s sign



How to: take any number from the interval, substitute it into the first differentiation and check the sign of the result (the value itself is not important, only the sign). The **positive** sign means **increasing** part of given function, the **negative** sign means **decreasing** part

$$f'(x)=\frac{2x^2\left(x^2-3\right)}{\left(x^2-1\right)^2}$$

$a)\ -2\in\left(-\infty,-\sqrt{3}\right)\rightarrow f'(-2)>0$ **increasing**

$b)\ -1\in\left(-\sqrt{3},-1\right)\rightarrow f'(-1)<0$ **decreasing**

$c)\ \frac{1}{2}\in\left(-1,1\right)\rightarrow f'\left(\frac{1}{2}\right)<0$ **decreasing**

$d)\ \frac{3}{2}\in\left(1,\sqrt{3}\right)\rightarrow f'\left(\frac{3}{2}\right)<0$ **decreasing**

$e)\ 2\in\left(\sqrt{3},\infty\right)\rightarrow f'(2)>0$ **increasing**

4. Second differentiation

$$f(x)=\frac{2x^3}{x^2-1} \quad \rightarrow \quad f'(x)=\frac{2x^2\left(x^2-3\right)}{\left(x^2-1\right)^2} \quad \rightarrow \quad f''(x)=\frac{4x\left(x^2+3\right)}{\left(x^2-1\right)^3}$$

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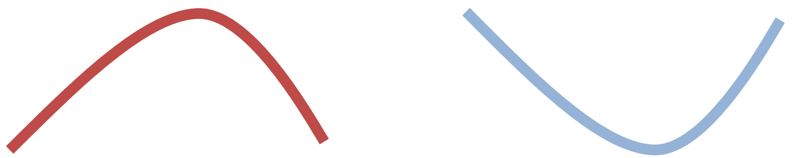
5. Curvature, inflexion points

Curvature – type of curve's shape – **concave** or **convex** ("hill" or "hole")

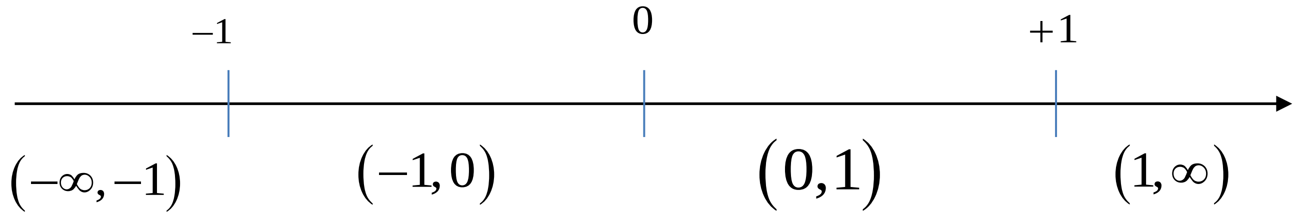


Inflexion point – point, where the type of **curvature is changing**. If there is an inflexion point, the **second differentiation is zero** there.

$$f''(x)=\frac{4x\left(x^2+3\right)}{\left(x^2-1\right)^3}=0 \Rightarrow x=0$$ - could be inflexion point, if the curvature changes in it.

5. Curvature, inflexion points

PoDs and f''(x) = 0 "divide" the real axis into intervals, where the curvature is studied with help of the second differentiation's sign



How to: take any number from the interval, substitute it into the second differentiation and check the sign of the result (the value itself is not important, only the sign). The **positive** sign means **convex** part of given function, the **negative** sign means **concave** part

$$f''(x)=\frac{4x\left(x^2+3\right)}{\left(x^2-1\right)^3}$$

$a)\ -2\in(-\infty,-1)\rightarrow f''(-2)<0$ **concave**

$b)\ -\frac{1}{2}\in(-1,0)\rightarrow f''\left(-\frac{1}{2}\right)>0$ **convex**

$c)\ \frac{1}{2}\in(0,1)\rightarrow f''\left(\frac{1}{2}\right)<0$ **concave**

$d)\ 2\in(1,\infty)\rightarrow f''(2)>0$ **convex**

x = 0 is the inflexion point (curvature is changed in it)

6. Local extrema

How to: take stationary points and substitute it into the second differentiation and check the sign of the result (the value itself is not important, only the sign). The **positive** sign means **local minimum** of given function, the **negative** sign means **local maximum**

$$f''(x) = \frac{4x\left(x^2+3\right)}{\left(x^2-1\right)^3} \qquad \text{SP} \rightarrow \begin{bmatrix} SP_1 = 0 \\ SP_2 = -\sqrt{3} \\ SP_1 = \sqrt{3} \end{bmatrix}$$

$f''\left(SP_1\right) = 0$ x = 0 is the inflexion point – so there is no extreme

$f''\left(SP_2\right) < 0$ **local maximum** $\rightarrow f\left(SP_2\right) = -3\sqrt{3} \doteq -5.2$

$f''\left(SP_3\right) > 0$ **local minimum** $\rightarrow f\left(SP_3\right) = 3\sqrt{3} \doteq 5.2$

7. Asymptotes

- lines (y = kx + q) the graph is “tending to in infinity” but does not cross it, nor touch it

Two options: a) asymptote without slope

b) asymptote with slope

asymptote without slope – means that the slope is “infinity”, what means the line is normal to the horizontal axis (or parallel to vertical axis of cartesian coordinate system)

asymptote with slope – means that the slope is “number”, there is a “angle” between line and horizontal axis

How to: asymptote without slope: line parallel to vertical axis passing PoDs

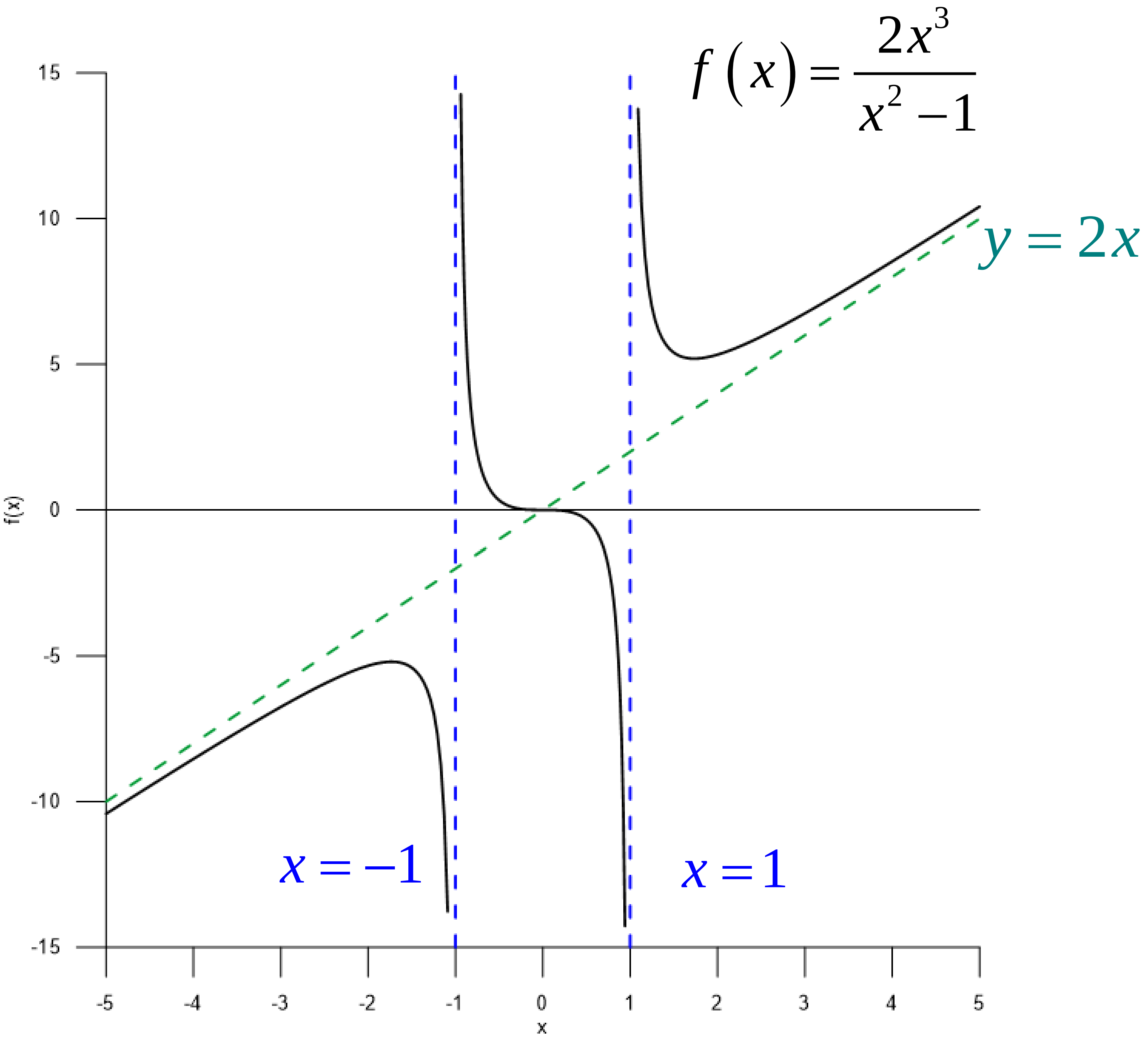
How to: asymptote with slope: the equation of the line is required – we need slope *k*, and absolute term *q:*

$$y = kx + q \quad \rightarrow \quad k = \lim_{x \to \pm\infty} \frac{f(x)}{x}; \qquad q = \lim_{x \to \pm\infty} \left[f(x) - kx \right]$$

$$k = \lim_{x \to \pm\infty} \frac{\frac{2x^3}{x^2 - 1}}{x} = \lim_{x \to \pm\infty} \frac{2x^3}{x(x^2 - 1)} = 2 \qquad q = \lim_{x \to \pm\infty} \left(\frac{2x^3}{x^2 - 1} - 2x \right) = \lim_{x \to \pm\infty} \left(\frac{2x^3 - 2x(x^2 - 1)}{x^2 - 1} \right) = 0$$

$$y = kx + q \quad \rightarrow y = 2x$$

8. Graph



9. Range and other properties

How to: Range: just from the graph

$$H(f) = \mathbb{R}$$

How to: Even/Odd: by definition

$$f(x) = -f(-x)$$ **Odd**

How to: Sup/Inf: by range

$$\sup f(x), \inf f(x) \rightarrow \nexists$$

Example 2

$$f(x)=\sqrt[3]{x^3+x^2}$$

1. Domain of definition, roots, points of discontinuity

$D(f)=\mathbb{R}$

$f(x)=0 \rightarrow x^3+x^2=0 \quad \Rightarrow \quad \text{roots:} \quad x_1=-1; \quad x_2=0$

$\text{PoD} \rightarrow \nexists$

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2. First differentiation, stationary points $f(x)=\sqrt[3]{x^3+x^2} \quad \rightarrow \quad f'(x)=\frac{1}{3}\cdot\frac{(3x^2+2x)}{(x^3+x^2)^{\frac{2}{3}}}$

Stationary point (SP) $\quad \text{SP} \rightarrow f'(x)=0 \Rightarrow \frac{1}{3}\cdot\frac{(3x^2+2x)}{(x^3+x^2)^{\frac{2}{3}}}=0 \Rightarrow SP=-\frac{2}{3}$

Warning: The first differentiation has the points of discontinuity (PoDs) which <u>are not the same</u> as those PoDs of original function (original function does not have any). You must take these points "into account" as well – when creating intervals of monotonicity (or curvature's intervals)

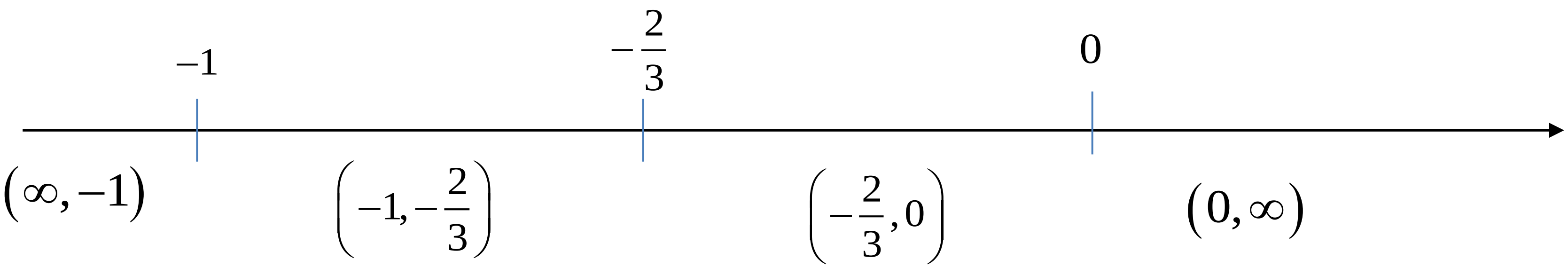
Points of discontinuity of the first differentiation (PoDoFD)

$$PoDoFD_1=-1$$

$$PoDoFD_2=0$$

If the first differentiation does not exist in the given point, there could be **sharp edge/peak,** or the function is **parallel to vertical axis**

3. Monotonicity



$$f'(x) = \frac{1}{3} \cdot \frac{(3x^2 + 2x)}{(x^3 + x^2)^{\frac{2}{3}}}$$

$a)(-\infty, -1) \rightarrow f' > 0$ **increasing**

$b)\left(-1, -\frac{2}{3}\right) \rightarrow f' > 0$ **increasing**

$c)\left(-\frac{2}{3}, 0\right) \rightarrow f' < 0$ **decreasing**

$d)(0, \infty) \rightarrow f' > 0$ **increasing**

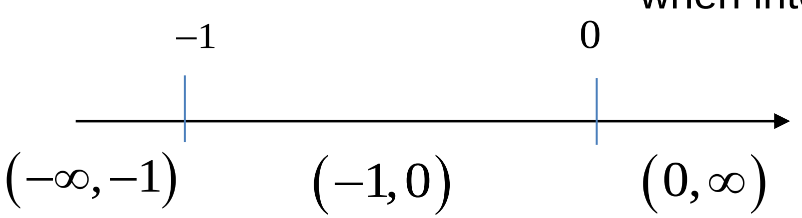
4. Second differentiation

$$f''(x) = -\frac{2x^2}{9\left(x^3 + x^2\right)^{\frac{5}{3}}}$$

5. Curvature, inflexion points

$$f''(x) = -\frac{2x^2}{9\left(x^3 + x^2\right)^{\frac{5}{3}}} = 0$$

Warning: It “seems”, that solution is x = 0, but it is the Point of Discontinuity of the Second Differentiation (PoDoSD) – again, these PoDoSDs are different (the second one is x = -1) then those of original function - we have to take them into account when intervals of curvature are created.



$a)\ (-\infty, -1) \rightarrow f'' > 0$ **convex**

$b)\ (-1, 0) \rightarrow f'' < 0$ **concave**

$c)\ (0, \infty) \rightarrow f'' > 0$ **convex**

x = -1 and x = 0 are the inflexion points

6. Local extrema

$$f''(x) = -\frac{2x^2}{9\left(x^3+x^2\right)^{\frac{5}{3}}}$$

$$f''(SP) = f''\left(-\frac{2}{3}\right) < 0$$ **local maximum**

Warning: We must "somehow" check the PoDoFD as well – there could be peaks/local extreme. We need the one-sided limits of the first differentiation in these points

$$\lim_{x\to -1^-} \frac{1}{3}\cdot\frac{\left(3x^2+2x\right)}{\left(x^3+x^2\right)^{\frac{2}{3}}} = \infty \qquad \lim_{x\to -1^+} \frac{1}{3}\cdot\frac{\left(3x^2+2x\right)}{\left(x^3+x^2\right)^{\frac{2}{3}}} = \infty$$

These limits are same, the function is parallel to vertical axis in the close vicinity of the point x = -1

$$\lim_{x\to 0^-} \frac{1}{3}\cdot\frac{\left(3x^2+2x\right)}{\left(x^3+x^2\right)^{\frac{2}{3}}} = -\infty \qquad \lim_{x\to 0^+} \frac{1}{3}\cdot\frac{\left(3x^2+2x\right)}{\left(x^3+x^2\right)^{\frac{2}{3}}} = \infty$$

These limits are not the same, the function has a peak in the point x = 0 – it is a local minimum

7. Asymptotes

asymptote without slope – there are none, there are not the PoDs

asymptote with slope:

$$k = \lim_{x\to\pm\infty} \frac{\sqrt[3]{x^3+x^2}}{x} = \boxed{\text{look and see method}} = 1$$

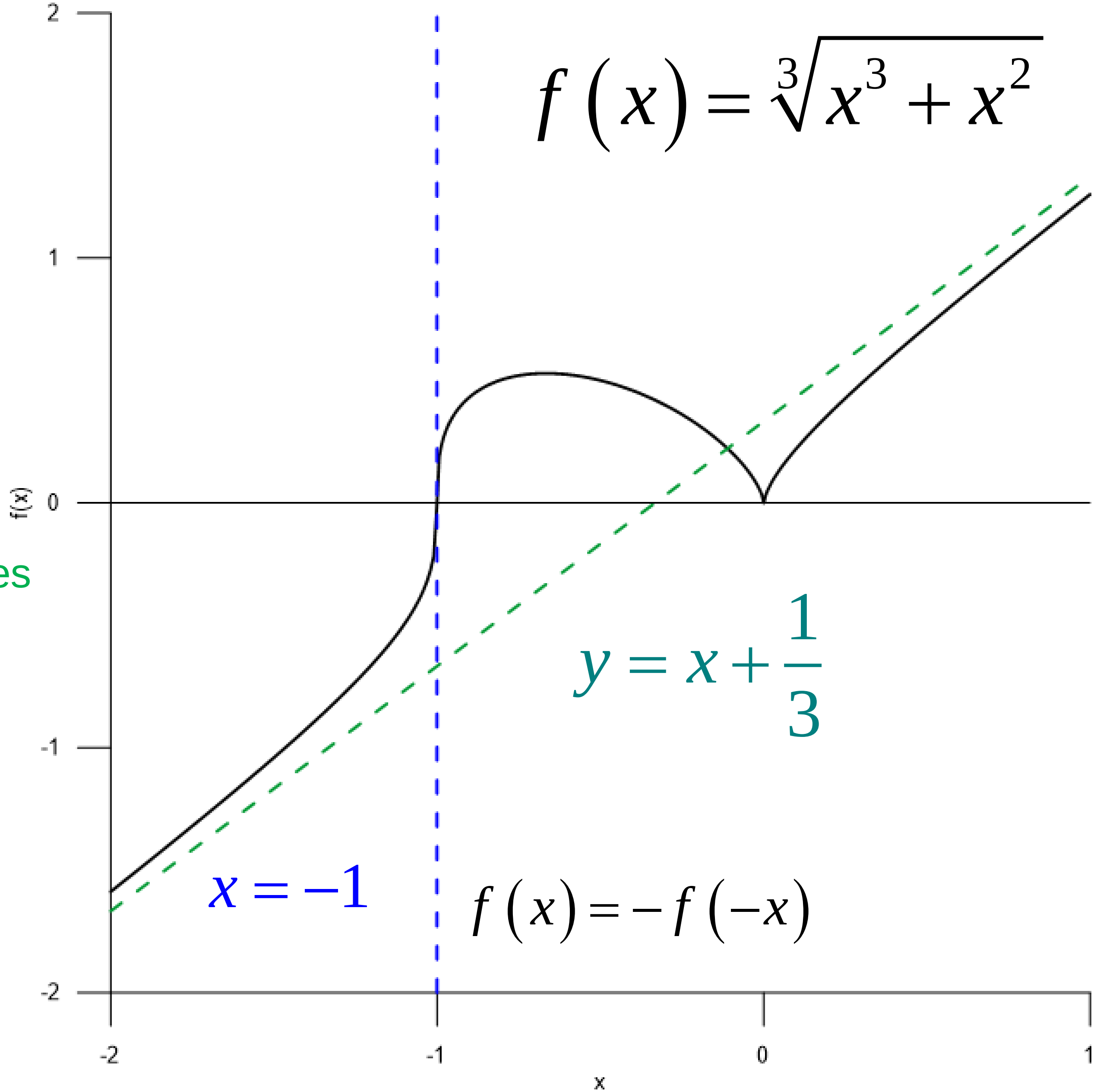
$$q = \lim_{x\to\pm\infty}\left(\sqrt[3]{x^3+x^2} - x\right) = \lim_{x\to\pm\infty}\left(\left(x^3+x^2\right)^{\frac{1}{3}} - x\right)\cdot \boxed{\frac{\left(x^3+x^2\right)^{\frac{2}{3}} + \left(x^3+x^2\right)^{\frac{1}{3}} + x^2}{\left(x^3+x^2\right)^{\frac{2}{3}} + x\left(x^3+x^2\right)^{\frac{1}{3}} + x^2}}_{=1} =$$

$$= \lim_{x\to\pm\infty} \frac{x^2}{\left(x^3+x^2\right)^{\frac{2}{3}} + x\left(x^3+x^2\right)^{\frac{1}{3}} + x^2} = \boxed{\text{look and see method}} = \frac{1}{3}$$

$$y = x + \frac{1}{3}$$

8. Graph

The function is parallel to the vertical axis close to the point x =-1 and there is a peak in the x = 0



9. Range and other properties

$$H(f)=\mathbb{R}$$