Mathematics for Biochemistry

LECTURE 10

Integration 1

Content:

- Introduction to integration
- Indefinite integration
- “Look and see” method
- Substitution method
- *Per partes* method
- Expansion, elementary fractions...

Integration – introduction

Integration is an **opposite** operation to **differentiation**

Integration - summation of infinitesimally small parts of the function *f*(*x*).

First fundamental theorem of calculus:

Indefinite integration of a function *f*(*x*) is related to its antiderivative.

We recognize two main types of integrals:

a) Indefinite integral – the result is **a function** $\int f(x)\,dx$

b) Definite integral– the results is a **number** $\int_a^b f(x)\,dx$

Notation: The integral sign $\int$ (modified S for “Sum”) represents integration. The symbol *dx*, called the differential of the variable *x*, indicates that the variable of integration is *x*. The function *f*(*x*) to be integrated is called the integrand. If a function can be integrated, it is said to be integrable – however this does not mean that integral exists. The points *a* and *b* are the endpoints/limits/boundaries of the integral.

Example 1

$$f(x) = x^2 \quad \rightarrow f'(x) = 2x$$

If the question is: what is the integration of f(x) = 2x ?

The **integration** and **differentiation** are in the **“direct/inverse” relationship**

$$f'(x) = 2x \quad \rightarrow \int f(x)\,dx = x^2$$

The **integration** of **given function** is **“something”** what we have **to differentiate** to obtain the **given function**

“Symbolic thinking”

$$f'(x) = \frac{df}{dx} \quad \rightarrow \quad f'(x)\,dx = df$$

$$\sum f'(x)\,dx \rightarrow \int f'(x)\,dx = f$$

Summation of **“plenty” (very) small** parts

Integration – introduction

The indefinite integral/antiderivative of a function ***f(x)*** is function as well, signed ***F(x)*** - also called as primitive function (despite the fact, that search for this "primitive function" is everything but primitive).

1. In the case of an **indefinite integral** we can write: $\int f(x)\,dx = F(x)$

so, it is valid: $F'(x) = f(x)$

but at the same time: $\left(F(x)+c\right)' = f(x),$ where *c* is a constant, $c \in$ **R**

c - constant of integration

so, in general: $\int f(x)\,dx = F(x) + c$

2. In the case of a **definite integral** we can write: $\int_a^b f(x) = F(b) - F(a)$

and constant of integration is erased.

Integration – calculating integrals

It is always important to write in the result of indefinite integration the arbitrary integration constant C, because without it confusing situations can occur, e.g.:

$$\int 2\sin x\cos x\,dx = \sin^2 x$$

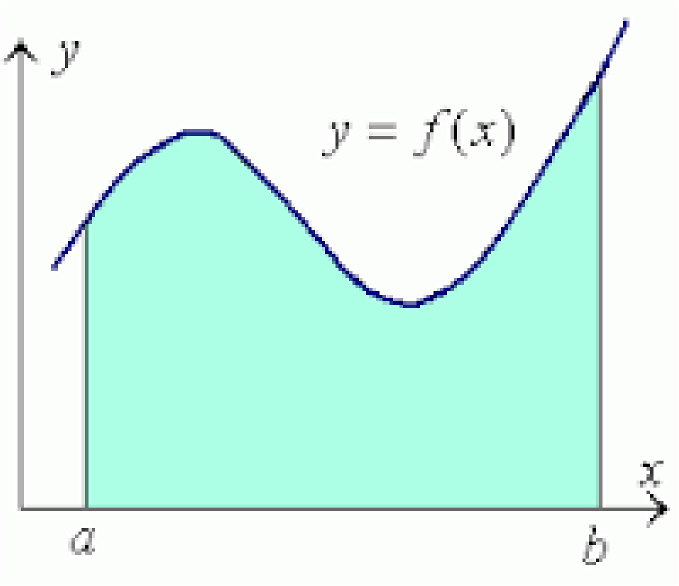
$$\int 2\sin x\cos x\,dx = -\cos^2 x$$

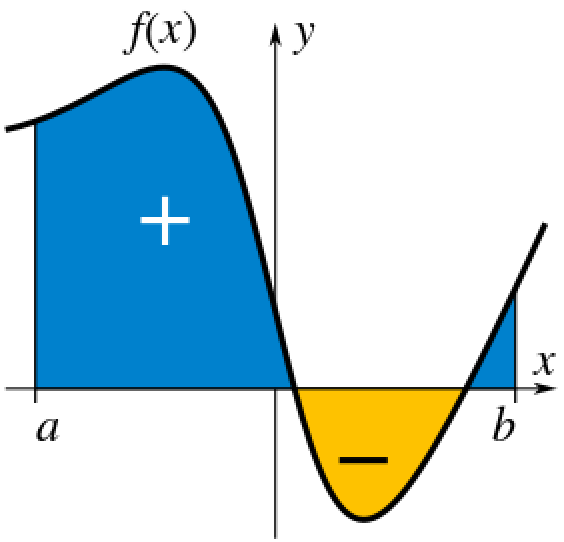
because: $\sin^2 x = -\cos^2 x + 1$

from it follows that in the first integral $C = -1$ and in the second $C = 1$.

To avoid this kind of confusion we will from now on never forget to include the "arbitrary constant $+C$" in our answer when we compute an antiderivative.

"Geometrical representation"





Definite integral of function *f*(*x*) on the interval <*a*,*b*> corresponds to the size of the area between the graph of function *f*(*x*) and the horizontal x-axis (limited by limits ***a*** and ***b***).

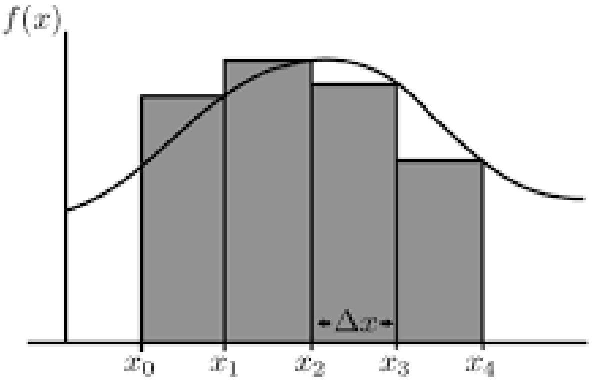
Important: The area can be also negative – it is dependent on the signs of functional values of the integrated function *f*(*x*).

This aspect of integration has many kinds of application in applied mathematics, physics and technics (and not only in 1 D!).

Integration – introduction

A simple "definition" of the definite integral is based on the limit of a Riemann sum of right rectangles. The exact area under a curve between *a* and *b* is given by the definite integral, which is defined as follows:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i)\cdot\left(\frac{b-a}{n}\right)\right]$$



$$dx \approx \Delta x = \frac{b-a}{n}$$

To make the approximating region you choose a *partition* of the interval $[a, b]$, i.e. you pick numbers $x_1 < \cdots < x_n$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These numbers split the interval $[a, b]$ into n sub-intervals

$$[x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{n-1}, x_n]$$

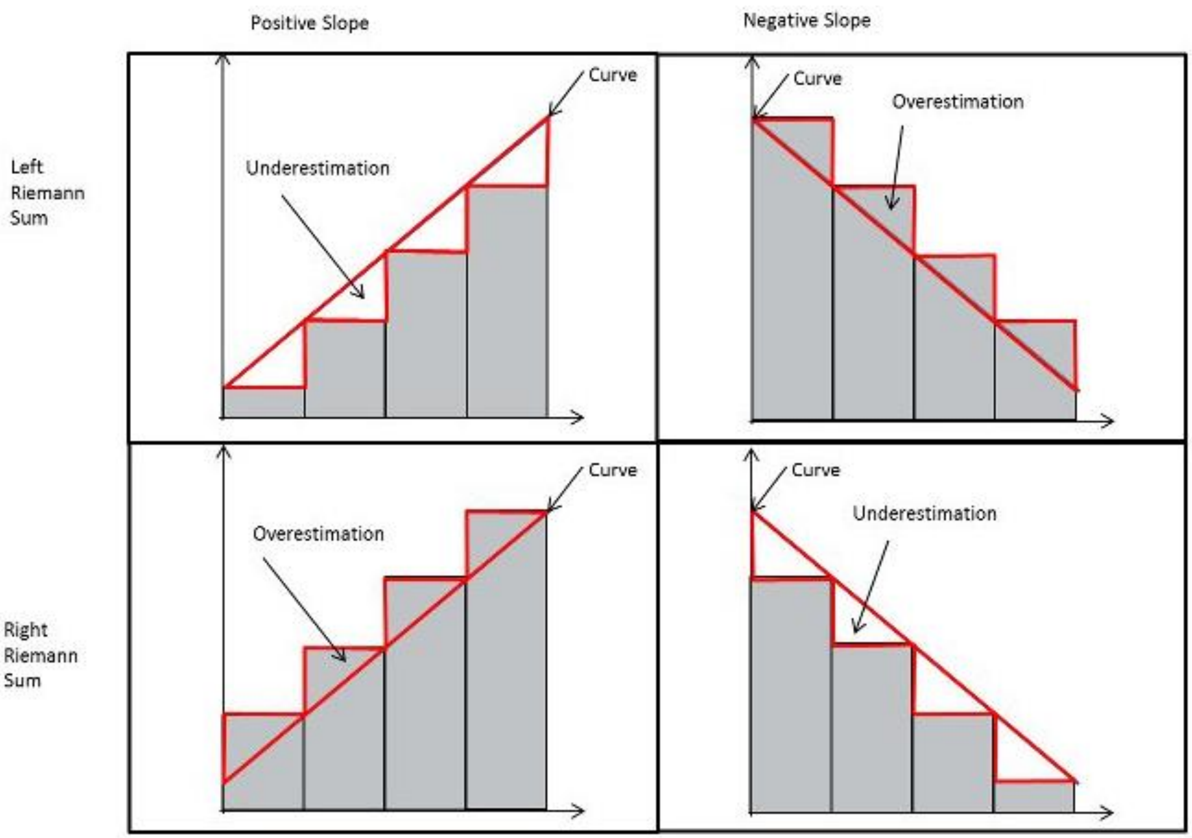
Integration – introduction

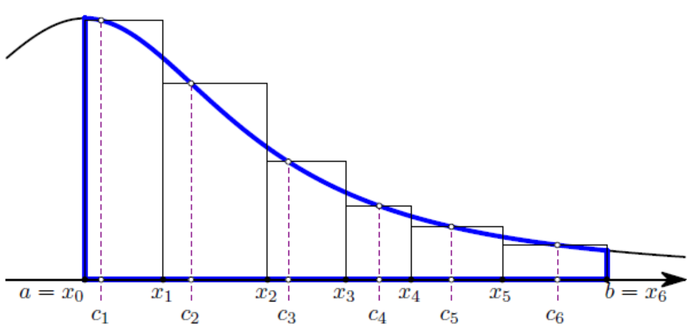
Limit of a Riemann sum of right rectangles:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i)\cdot\left(\frac{b-a}{n}\right)\right]$$

There can originate 3 different situations:

- so-called left Riemann sum,
- so-called right Riemann sum,
- and so-called midpoint Riemann sum.





midpoint Riemann sum

Integration – calculating integrals

Indefinite integral	Definite integral
$\int f(x)dx$ is a function of x.	$\int_a^b f(x)dx$ is a number.
By definition $\int f(x)dx$ is *any function of x whose derivative is $f(x)$.*	$\int_a^b f(x)dx$ was defined in terms of Riemann sums and can be interpreted as "area under the graph of $y = f(x)$", at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2xdx = x^2 + C$ and $\int 2tdt = t^2 + C$ are functions of different variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2xdx = 1$, and $\int_0^1 2tdt = 1$, so $\int_0^1 2xdx = \int_0^1 2tdt$.

Integration – calculating integrals

Basic „rule“ for the solution of indefinite integrals is:

„perfect knowledge of differentiation!!!“

(integration of basic functions is simply an opposite operation to the differentiation).

But there exist few methods, which can help...

It is good always to check the result of integration – by means of differentiation (as a test).

Important! Not every integral can be solved – in contrary to the differentiation, where almost all functions can be differentiated.

Integration – calculating integrals

Integration – "look and see" – based on the differentiation tables

$$(c)' = 0 \quad \rightarrow \quad \int 0 dx = c$$

$$\left(x^n\right)' = nx^{n-1} \quad \rightarrow \quad \int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$$

$$(\ln x)' = \frac{1}{x} = x^{-1} \quad \rightarrow \quad \int \frac{1}{x} dx = \ln |x| + c$$

$$\left(e^x\right)' = e^x \quad \rightarrow \quad \int e^x dx = e^x + c$$

$$(\sin x)' = \cos x \quad \rightarrow \quad \int \cos x dx = \sin x + c$$

$$(\cos x)' = -\sin x \quad \rightarrow \quad \int \sin x dx = -\cos x + c$$

$$(\tan x)' = \frac{1}{\cos^2 x} \quad \rightarrow \quad \int \frac{1}{\cos^2 x} dx = \tan x + c$$

$$(\arctan x)' = \frac{1}{1+x^2} \quad \rightarrow \quad \int \frac{1}{1+x^2} dx = \arctan x + c$$

Integration – calculating integrals

Integration – “look and see” – based on the differentiation tables

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad \to \quad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c = -\arccos x + c$$

$$\big(f(x)+g(x)\big)' = f'(x)+g'(x) \quad \to \quad \int \big[f(x)+g(x)\big]dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\big(cf(x)\big)' = cf'(x) \quad \to \quad \int cf(x)\,dx = c\int f(x)\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

Example 2

$$\int 7x^4 dx = 7\int x^4 dx = 7\frac{x^{4+1}}{4+1} + c = \frac{7}{5}x^5 + c; \qquad \int \frac{3}{1+x^2}\,dx = 3\arctan x$$

$$\int \left(-\frac{3}{\sqrt{1-x^2}} + \frac{1}{2x}\right)dx = -3\int \frac{1}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int \frac{1}{x}\,dx = 3\arccos x + \frac{1}{2}\ln|x| + c$$

$$\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$$

Integration – calculating integrals

Integration – substitution method $\int f\left(g\left(x\right)\right)g'\left(x\right)dx=\int f\left(t\right)dt \qquad t=g\left(x\right)$

Example 3

$$\int x\cdot e^{x^2}dx = /t=e^{x^2} \rightarrow dt=\left(e^{x^2}\right)'dx = 2x\cdot e^{x^2}dx \Rightarrow dx=\frac{dt}{2x\cdot e^{x^2}}/ = \int x\cdot t\frac{dt}{2x\cdot \underbrace{e^{x^2}}_{t}} =$$

$$=\int t\frac{dt}{2t}=\frac{1}{2}\int 1dt=\frac{t}{2}+c=\frac{1}{2}e^{x^2}+c$$

Example 4

$$\int\frac{b}{\left(x-a\right)^2+b^2}dx=b\int\frac{dx}{\left(x-a\right)^2+b^2}=b\int\frac{dx}{b^2\left[\frac{\left(x-a\right)^2}{b^2}+1\right]}=\frac{1}{b}\int\frac{dx}{\left(\frac{x-a}{b}\right)^2+1}=$$

$$=\begin{bmatrix} t=\left(x-a\right)/b \\ dt=dx/b \\ dx=bdt \end{bmatrix}=\frac{1}{b}\int\frac{bdt}{t^2+1}=\int\frac{1}{t^2+1}dt=artctg\left(t\right)+C=artctg\left(\frac{x-a}{b}\right)+C$$

Integration – calculating integrals

Integration – substitution method $\int f\left(g\left(x\right)\right)g'\left(x\right)dx=\int f\left(t\right)dt$ $t=g\left(x\right)$

Example 5

$$\int \frac{\sqrt{1+\ln x}}{x}dx = /t=1+\ln x \rightarrow dt=\frac{1}{x}dx \Rightarrow dx=xdt/ =$$

$$=\int \sqrt{t}dt=\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{3}\left(1+\ln x\right)^{\frac{3}{2}}+c$$

Example 6

$$\int \frac{x^2}{\sqrt{1-x^6}}dx = /t=x^3 \rightarrow dx=\frac{1}{3x^2}dt/ = \int \frac{x^2}{3x^2\sqrt{1-t^2}}dt=\frac{1}{3}\int \frac{1}{\sqrt{1-t^2}}dt=$$

$$=\frac{1}{3}\arcsin t+c=\frac{1}{3}\arcsin x^3+c$$

Integration – calculating integrals

Integration – "per partes" method $$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

$$\int u'v\,dx = uv - \int uv'dx$$

Example 7

$$\int x\sin x\,dx = \begin{bmatrix} u' = \sin x & v = x \\ u = -\cos x & v' = 1 \end{bmatrix} = -x\cos x - \int 1\cdot(-\cos x)\,dx = -x\cos x + \sin x + c$$

Example 8

$$\int \ln x\,dx = \begin{bmatrix} u' = 1 & v = \ln x \\ u = x & v' = \dfrac{1}{x} \end{bmatrix} = x\ln x - \int \frac{x}{x}\,dx = x\ln x + x + c = x(\ln x + 1) + c$$

Example 9

$$\int x^2\sin x\,dx = \begin{bmatrix} u' = \sin x & v = x^2 \\ u = -\cos x & v' = 2x \end{bmatrix} = -x^2\cos x + \int 2x\cos x\,dx = \begin{bmatrix} u' = \cos x & v = x \\ u = \sin x & v' = 1 \end{bmatrix} =$$

$$= -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) = -x^2\cos x + 2x\sin x - 2\cos x + c$$

Integration – calculating integrals

Integration – expansion, elementary fractions

Example 10

$$\int \frac{1}{x(x+1)}dx = \left[\begin{array}{l} \dfrac{1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1} = \dfrac{A(x+1)+Bx}{x(x+1)} = \dfrac{x(A+B)+A}{x(x+1)} \\ \Rightarrow x(A+B)+A = 1 \Rightarrow A = 1, B = -1 \end{array} \right] =$$

$$= \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = \ln|x| - \ln|x+1| + c$$

Example 11

$$\int \frac{x^4}{x^2+1}dx = \left[\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1} \right] = \int \left(x^2 - 1 + \frac{1}{x^2+1} \right) dx = \frac{x^3}{3} - x + \arctan x + c$$

Integration – calculating integrals

Integration – further examples

$$\int e^x \sin x\,dx = \begin{bmatrix} u' = e^x & v = \sin x \\ u = e^x & v' = \cos x \end{bmatrix} = e^x \sin x - \int e^x \cos x\,dx = \begin{bmatrix} u' = e^x & v = \cos xx \\ u = e^x & v' = -\sin x \end{bmatrix} =$$

$$= e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right) = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\,dx}_{I}$$

$$\Rightarrow I = e^x\left(\sin x - \cos x\right) - I \Rightarrow 2I = e^x\left(\sin x - \cos x\right) \Rightarrow$$

$$I = \int e^x \sin x\,dx = \frac{1}{2} e^x \left(\sin x - \cos x\right) + c$$

$$\int \frac{4}{\cos x}\,dx = 4\int \frac{\cos x}{\cos^2 x}\,dx = 4\int \frac{\cos x}{1-\sin^2 x}\,dx = \begin{bmatrix} \sin x = t \\ \cos x\,dx = dt \end{bmatrix} = 4\int \frac{\cos x}{\left(1-t^2\right)\cos x}\,dt =$$

$$= 4\int \frac{1}{1-t^2}\,dt = \left[\frac{1}{1-t^2} = \frac{1}{2}\cdot\frac{1}{1-t} + \frac{1}{2}\cdot\frac{1}{1+t}\right] = 4\int \left(\frac{1}{2}\cdot\frac{1}{1-t} + \frac{1}{2}\cdot\frac{1}{1+t}\right)dt =$$

$$= 2\left[\int \frac{1}{1-t}\,dt + \int \frac{1}{1+t}\,dt\right] = 2\left(-\ln\left|1-t\right| + \ln\left|1+t\right|\right) = 2\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c$$

Non-elementary integrals.

Can not be solved in real numbers domain with the help of these simple rules. Some of them can be solved with the help of Taylor series or with complex number functions.

$$\sin x^2, \cos x^2, e^{x^2}, \ln(\ln x), \sqrt{1-x^4}, \frac{e^x}{x}, \cdots$$

Such functions are so-called **integrable** (they fulfil the conditions), but their indefinite integrations (primitive functions) often do not exist. However, some of their definite integrations exist, e.g.:

$$\int_0^\infty \sin x^2 dx = \sqrt{\frac{\pi}{2}}, \qquad \int_{-\infty}^{\infty} e^{-x^2} dx = 1$$