Mathematics for Biochemistry

LECTURE 11

Integration 2

Content:

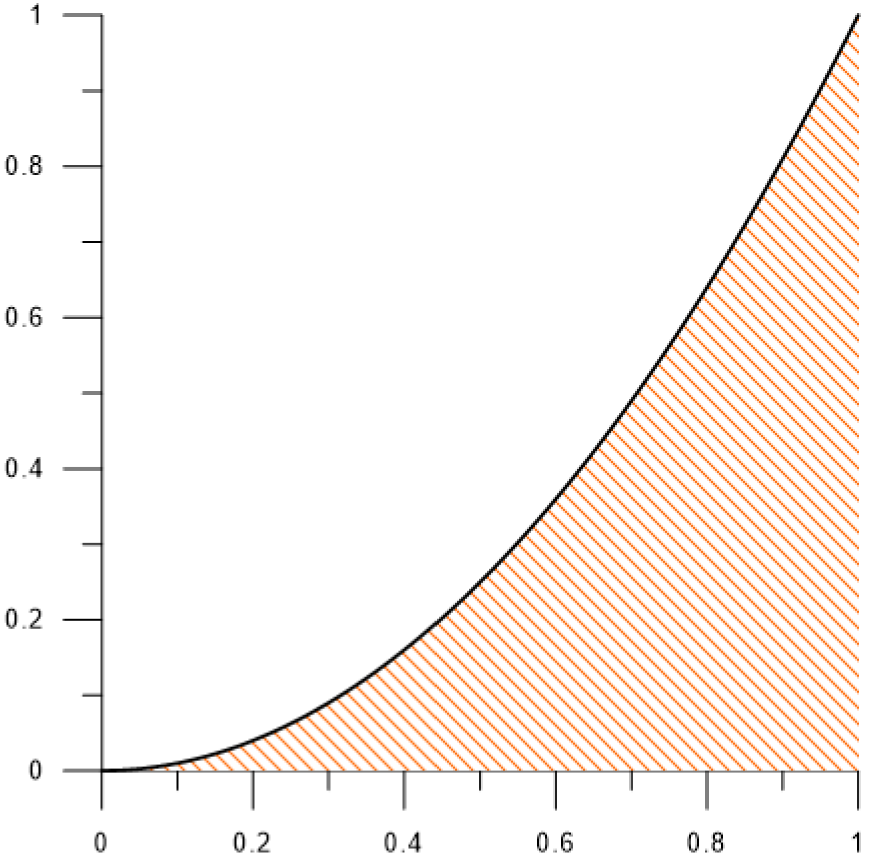
- Definite integration
- Substitution method
- *Per partes* method
- Properties
- Applications
- Improper integrals

Definite Integration – simple examples

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example 1.

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\left(1^3 - 0^3\right) = \frac{1}{3}$$

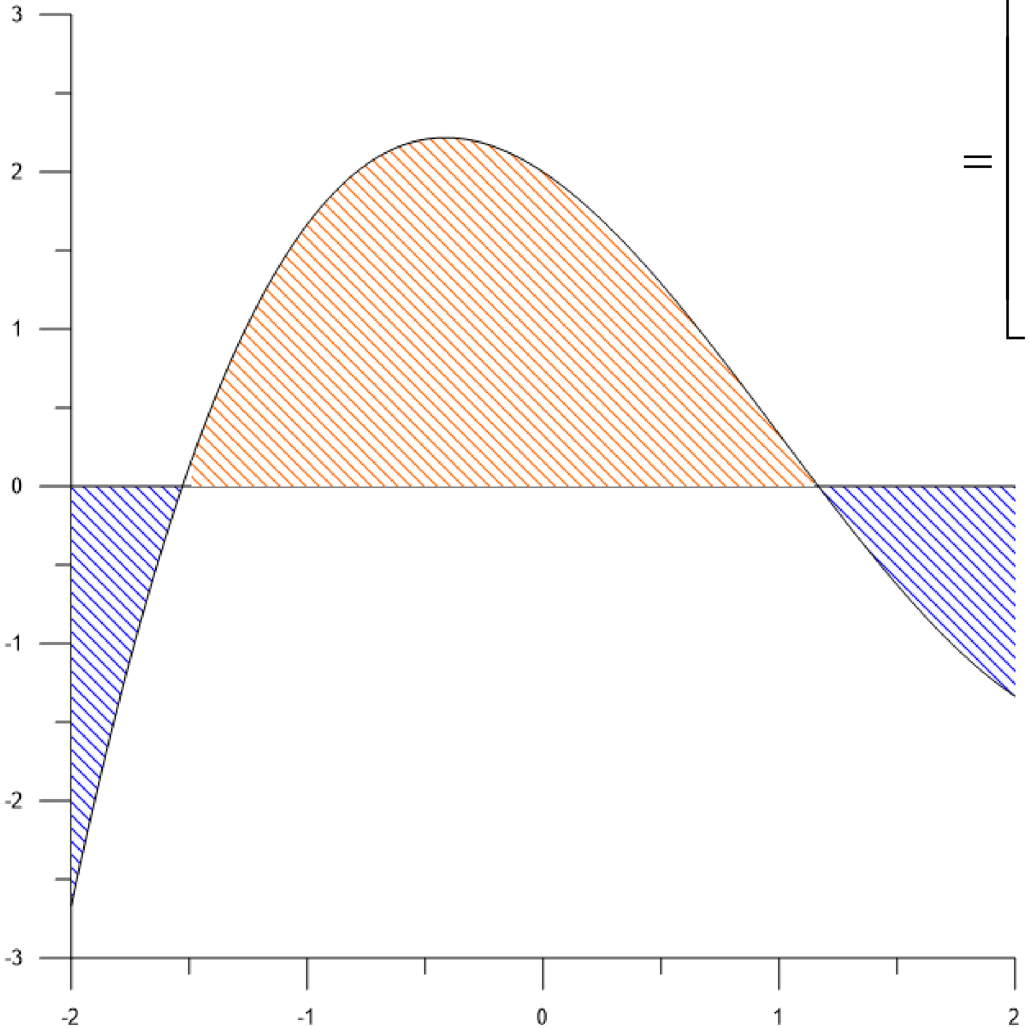


Definite Integration – simple examples

Example 2.

$$\int_{-2}^{2}\left(\frac{x^3}{3}-x^2-x+2\right)dx=\left[x^4-\frac{x^3}{3}-\frac{x^2}{2}+2x\right]_{-2}^{2}$$

$$=\left[\begin{array}{l}\left(2^4-\dfrac{2^3}{3}-\dfrac{2^2}{2}+2\cdot 2\right)- \\ \left((-2)^4-\dfrac{(-2)^3}{3}-\dfrac{(-2)^2}{2}+2\cdot(-2)\right)\end{array}\right]=\frac{8}{3}$$



Example 3. $$\int_{-2}^{-1} \frac{xe^x+1}{x}dx = \int_{-2}^{-1} e^x dx + \int_{-2}^{-1} \frac{1}{x}dx = \left[e^x + \ln|x|\right]_{-2}^{-1} =$$

$$= \left(\frac{1}{e}+0\right) - \left(\frac{1}{e^2}+\ln 2\right) = \frac{1}{e} - \frac{1}{e^2} - \ln 2$$

Warning!

$$\int_{-1}^{1} \frac{1}{x}dx = \left[\ln|x|\right]_{-1}^{1} = \ln|1| - \ln|-1| = 0$$

Wrong

Integration around Point of Discontinuity is not correct $\rightarrow 0 \in \langle -1, 1 \rangle$

Definite Integration – substitution method

Example 4.

$$\int_0^1 \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx = \left\| \begin{matrix} 1+e^{4x}=t & 4e^{4x}dx=dt \\ t(0)=1+e^{4\cdot 0}=2 & t(1)=1+e^{4\cdot 1}=1+e^4 \end{matrix} \right\| = \frac{1}{4}\int_2^{1+e^4}\frac{1}{\sqrt{t}}dt =$$

$$= \frac{1}{4}\left[2\sqrt{t}\right]_2^{1+e^4} = \frac{1}{2}\left(\sqrt{1+e^4}-\sqrt{2}\right)$$

or carry out indefinite integration, go back to original variable and use the original limits

$$\int_0^1 \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx \to \int \frac{e^{4x}}{\sqrt{1+e^{4x}}}dx = \left\| \begin{matrix} 1+e^{4x}=t \\ 4e^{4x}dx=dt \end{matrix} \right\| = \frac{1}{4}\int\frac{1}{\sqrt{t}}dx =$$

$$= \frac{1}{4}\left[2\sqrt{t}\right] \to \frac{1}{2}\left[\sqrt{1+e^{4x}}\right]_0^1 = \frac{1}{2}\left(\sqrt{1+e^4}-\sqrt{2}\right)$$

Definite Integration – per partes method

Example 5.

$$\int_0^{\frac{\pi}{2}} x\sin x dx = \left\| \begin{matrix} u' = \sin x & u = -\cos x \\ v = x & v = 1 \end{matrix} \right\| = \left[-x\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos x\right) dx =$$

$$= -\left[x\cos x\right]_0^{\frac{\pi}{2}} + \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$

Example 6.

$$\int_0^{\frac{\pi}{2}} x\sin x dx = \left\| \begin{matrix} u' = \sin x & u = -\cos x \\ v = x & v = 1 \end{matrix} \right\| = \left[-x\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos x\right) dx =$$

$$= -\left[x\cos x\right]_0^{\frac{\pi}{2}} + \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$

Definite Integration – properties

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad c \in \langle a, b \rangle$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \qquad f(x) \text{ is even function}$$

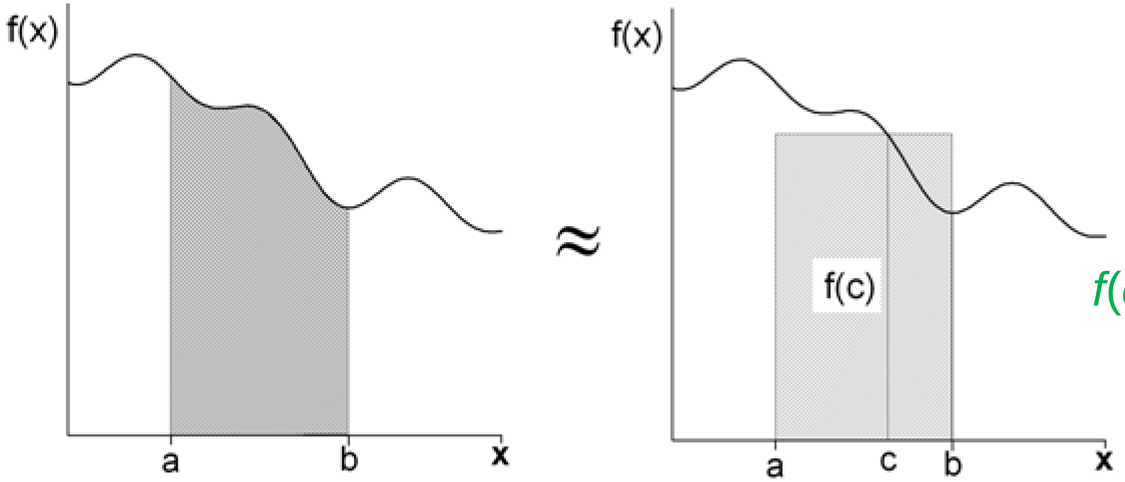
$$\int_{-a}^a f(x)\,dx = 0 \qquad f(x) \text{ is odd function}$$

Definite integrals – properties (mean value theorem)

First mean value theorem for definite integrals:

Let *f*(*x*) be a continuous function on <*a*, *b*>. Then there exists a point *c* in <*a*, *b*>, such that it is valid:

$$\int_a^b f(x)\,dx \approx f(c)(b-a)$$



f(*c*) – mean value

<u>Simply</u>: The area below the graph *f*(*x*) can be approximated by a rectangle with the height *f*(*c*) – and for continuous functions there always exists such a point *c* in <*a*, *b*> (... and we do not have to know its exact value).

Definite integrals – properties

(the definite integral as a function of its integration bounds)

What will happen when we set for the upper bound not a number, but a variable?

Consider the expression

$$I = \int_0^x t^2 \, dt.$$

What does I depend on?

To see this, you calculate the integral and you find

$$I = \left[\tfrac{1}{3}t^3\right]_0^x = \tfrac{1}{3}x^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3}x^3.$$

So the integral depends on x.

It does not depend on t, since t is a "dummy variable".

In this way you can use integrals to define new functions.

Such expressions are called as integrals as a function of its upper bound (in general: function as a function of its integration bounds).

The definite integral as a function of its integration bounds.

An example of a function defined by an integral is the "error-function" from statistics. It is given by

$$\operatorname{erf}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

so $\operatorname{erf}(x)$ is the area of the shaded region in figure.

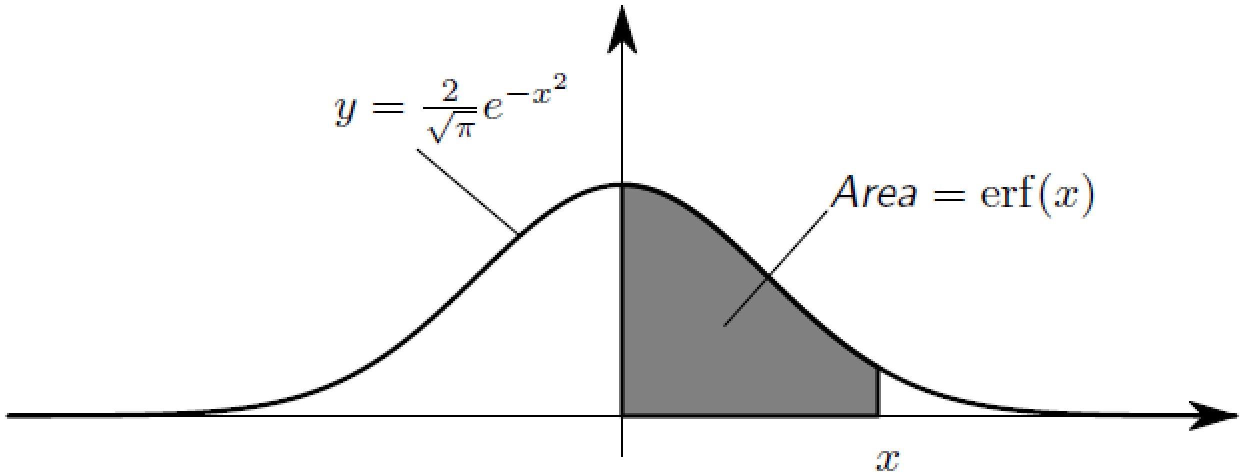


Figure. Definition of the Error function.

The integral cannot be computed in terms of the standard functions (square and higher roots, sine, cosine, exponential and logarithms).

Since the integral occurs very often in statistics it has been given a name, namely, "$\operatorname{erf}(x)$".

The definite integral as a function of its integration bounds.

An interesting question arises (?):

How do you differentiate a function that is defined by an integral?

The answer is simple

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\{F(x) - F(a)\} = F'(x) = f(x),$$

and therefore

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

Comment: For indefinite integrals it is clear that the differentiation of their result is the original function (it is an opposite operation). But this is not the case for definite integrals (because the results of them are numbers=constants); Only for the definite integral as a function of its integration bounds we can write this result.

Definite integrals – some applications

The length of curve (function's graph)

$$L=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

Volume of rotational body – rotation around horizontal axis

$$V_x=\pi\int_a^b f^2(x)\,dx$$

Volume of rotational body – rotation around vertical axis

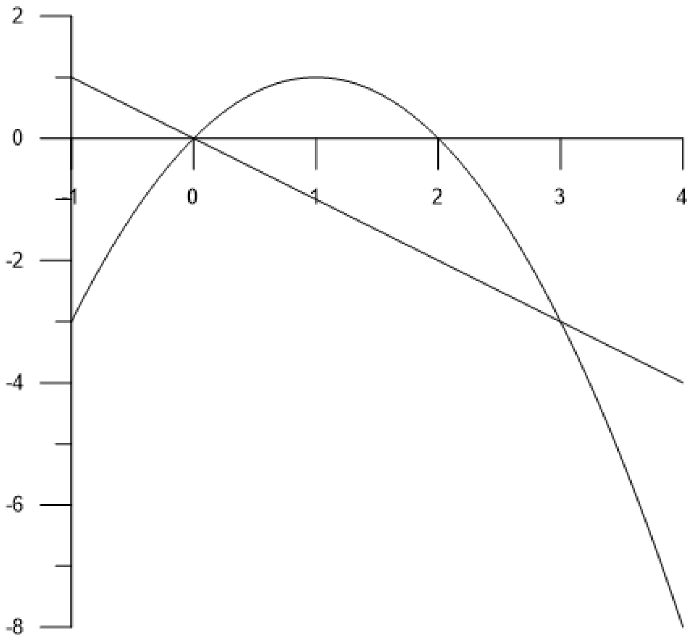
$$V_y=2\pi\int_a^b x\cdot f(x)\,dx$$

Area of rotational surface – rotation around horizontal axis

$$S_x=2\pi\int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

Example 7. Find the area of region bounded by two functions:

$$f_1(x) = -x \qquad f_2(x) = 2x - x^2$$



Boundaries are intersection points of given functions

$$\int_a^b \left[f_2(x) - f_1(x) \right] dx \rightarrow \int_0^3 \left[2x - x^2 - (-x) \right] dx = \int_0^3 \left[3x - x^2 \right] dx =$$

$$= \left[3\frac{x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{9}{2}$$

Example 8. Find the length of the given function on the interval <1,10>

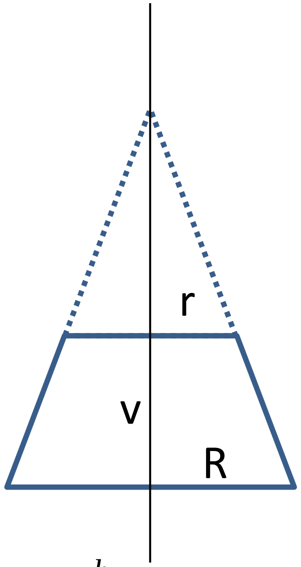
$$f\left(x\right)=\frac{x^2}{4}-\frac{1}{2}\ln x$$

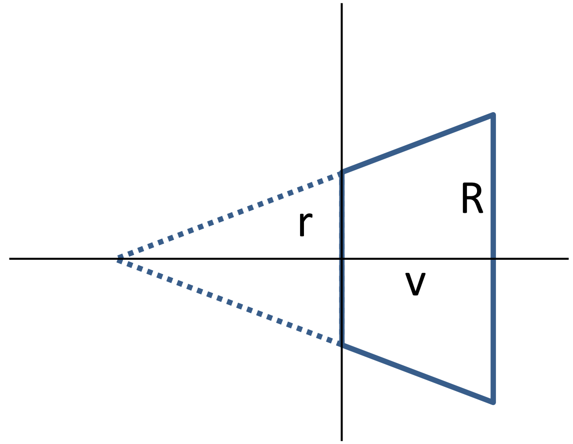
$$L=\int_a^b\sqrt{1+\left[f'\left(x\right)\right]^2}dx\to f'\left(x\right)=\frac{x}{2}-\frac{1}{2x}\to L=\int_1^{10}\sqrt{1+\left[\frac{x}{2}-\frac{1}{2x}\right]^2}dx=$$

$$=\int_1^{10}\sqrt{1+\left(\frac{x^2-1}{2x}\right)^2}dx=\int_1^{10}\sqrt{\frac{\left(x^2+1\right)^2}{\left(2x\right)^2}}dx=$$

$$=\int_1^{10}\frac{x^2+1}{2x}dx=\left[\frac{x^2}{4}+\frac{1}{2}\ln x\right]_1^{10}=\frac{99}{4}-\frac{1}{2}\ln 10$$

Example 9. Find the volume of the conical frustum





$$V = \pi \int_a^b f^2(x)\,dx \rightarrow y = kx + q; \quad a = 0, b = v \rightarrow$$

$$V = \int_0^v (kx+q)^2\,dx = \left[\frac{(kx+q)^3}{3k}\right]_0^v = \frac{(kv+q)^3}{3k} \rightarrow$$

$$\rightarrow k = \frac{R-r}{v}; \quad q = r \rightarrow V = \pi \frac{\left(\frac{R-r}{v}v + r\right)^3}{3\frac{R-r}{v}} = \pi \frac{R^3 v}{3(R-r)} = \frac{\pi v}{3}\left(R^2 + Rv + r^2\right)$$

Improper integrals

Integration on the unbounded interval

$$\int_a^{\infty} f(x)\,dx \quad \rightarrow \quad = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_a^b f(x)\,dx$$

if the function f(x) is unbounded or discontinuous on given interval

Improper integrals

Example 10.

$$\int_2^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_2^b = -\lim_{b\to\infty}\left[\frac{1}{b}-\frac{1}{2}\right] = \frac{1}{2}$$

Example 11.

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \lim_{b\to\infty}\left[\ln|b| - \ln|1|\right] = \nexists$$