

Mathematics for Biochemistry

LECTURE 12

Ordinary differential equations

DEFINITION

An equation containing the differentiation of one or more **dependent variables**, with respect to one or more **independent variables**

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

1. TYPE

2. ORDER

3. LINEARITY

TYPE

ORDINARY

Involve only **ONE**
independent variable

$$y' - 3y = e^{2x}$$

$$y'' + 2y' - 3y = 0$$

PARTIAL

Involve **TWO or MORE**
independent variable

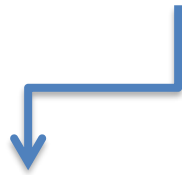
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

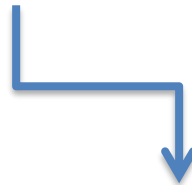
ORDER

The *order* of differential equation is the **order** of the **highest** differentiation

$$y'' + 2y' - 3y = 0$$



second order



first order

LINEARITY

The differential equation of order “n” is said to be **linear**, if the function

$$F\left(x, y', y'', \dots, y^{(n)}\right) = 0 \quad \text{or} \quad F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

is linear in the variables $y, y', \dots, y^{(n)}$

There are no multiplications of dependent variables and their differentiations.

$$y'' + 2x^3 y' - 3y = 0 \quad \text{– linear DE}$$

$$y'' + 2y \cdot y' - 3y = 0 \quad \text{– nonlinear DE}$$

LINEAR ODE

HOMOGENOUS

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \dots + p_2(x)y'' + p_1(x)y' + p_0(x)y(x) = 0$$

$$(1+x^3)y^{(5)} - y^{(4)}(x) + xy' + 3y = 0 \quad \text{where}$$

$$p_5 = 1+x^3, p_4 = -1, p_3 = 0, p_2 = 0, p_1 = x, p_0 = 3$$

IMPORTANT PROPERTY OF LINEAR DE

If $y_1(x)$ is first solution and $y_2(x)$ is the second solution, then the function $y_1(x) + a y_2(x)$ is solution, too (a is real number)

NON-HOMOGENOUS

$$y' + y + 2 = 0$$

ORIGIN OF DE

- PHYSICS (and all other natural sciences)
- ECONOMICS
- SOCIOLOGY

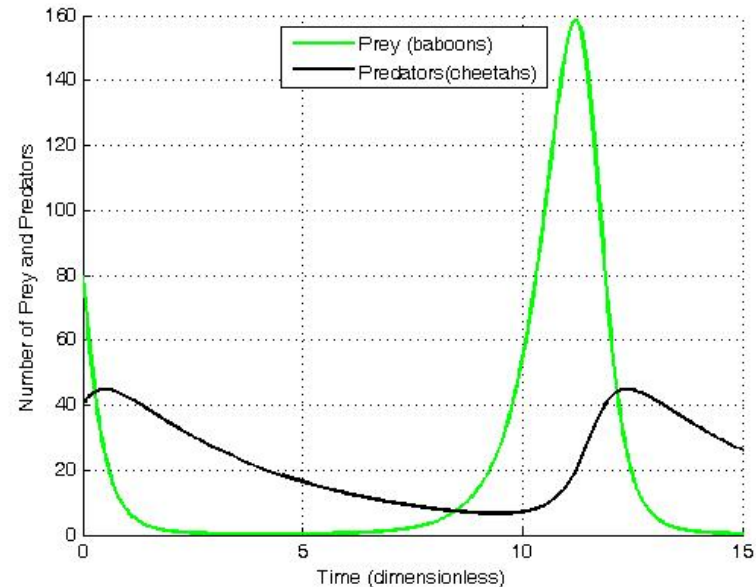
EXAMPLE FROM BIOLOGY

PREDATOR – PREY MODEL (Lotka – Volterra model)

$$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = \delta xy - \gamma y$$

where

- x is the **number of prey** (e.g. rabbits)
- y is the **number of predator** (e.g. foxes)
- dx/dt and dy/dt represent the **growth rates** of these two population over **time t**
- $\alpha, \beta, \gamma, \delta$ are positive real parameters describing the **interaction of the species**



SOLUTIONS OF ODE

ELEMENTARY SITUATION

$$\frac{dy}{dx} = y'(x) = f(x)$$

SOLUTION by simple integration: $y(x) = \int f(x) dx$

- generalization: $y^{(n)}(x) = f(x) \rightarrow y(x) = \underbrace{\int \int \dots \int}_{n \text{ times}} f(x) dx dx \dots dx$

IMPORTANT: The first solution (the DE of the first order) will contain **ONE** integration constant, because of **ONE** integration. The solution of generalized case will contain **n constants** because of **n** integrations. The final values of the constants are dependent on **boundary conditions**.

EXAMPLE 1

$$y'(x) = x - \sin x \quad \text{boundary condition:} \quad y(0) = -1$$

Solution

$$y(x) = \int (x - \sin x) dx = \frac{x^2}{2} + \cos x + C$$

The constant C is set through boundary condition:

$$y(0) = -1 \rightarrow \frac{0^2}{2} + \cos 0 + C = -1 \Rightarrow C = -2$$

The final solution is:

$$y(x) = \frac{x^2}{2} + \cos x - 2$$

EXAMPLE 2

$$y'''(x) = 1 - x^2$$

no boundary conditions

Solution

$$y(x) = \int \left(\int \left(\int (1 - x^2) dx \right) dx \right) dx = -\frac{x^5}{60} + \frac{x^3}{6} + C_1 \frac{x^2}{2} + C_2 x + C_3$$

EXAMPLE 3

$$y''(x) = x^2$$

boundary conditions: $y(0) = 0$; $y'(0) = 1$

Solution

$$y(x) = \int \left(\int x^2 dx \right) dx = \frac{x^4}{12} + C_1 x + C_2$$

$$y(0) = 0 \Rightarrow C_2 = 0; \quad y'(0) = 1 \Rightarrow C_1 = 1$$

SOLUTIONS OF ODE

DE of the 1st order in **SEPARABLE FORM**

$$y'(x) = f(x) \cdot g(y)$$

SOLUTION

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\frac{dy}{g(y)} = f(x) dx \quad \text{separation}$$

$$\int \frac{dy}{g(y)} = \int f(x) dx \quad \text{integration}$$

$$y(x) = \dots \quad \text{explicit form if possible}$$

EXAMPLE 1

$$y'(x) = -y^4$$

$$\text{boundary condition: } y(0) = 1$$

Solution

$$\frac{dy}{dx} = -y^4$$

$$\frac{dy}{y^4} = -dx \rightarrow \int \frac{dy}{y^4} = -\int dx$$

$$\frac{1}{-3y^3} = -x + C \Rightarrow y = \frac{1}{\sqrt[3]{3x + C}}$$

The constant C is: $C = 1$

The **final solution** is: $y(x) = \frac{1}{\sqrt[3]{3x + 1}}$

EXAMPLE 2

$$y'(x) = \frac{y}{y-1} x$$

boundary condition: $y(2) = 2$

Solution

$$\frac{dy}{dx} = \frac{y}{y-1} x$$

$$\frac{y-1}{y} dy = x dx \rightarrow \int \frac{y-1}{y} dy = \int x dx$$

$$y - \ln|y| = \frac{x^2}{2} + C \quad \text{can not be rewritten to explicit form}$$

The constant C is: $2 - \ln|2| = \frac{2^2}{2} + C \Rightarrow C = -\ln 2$

The **final solution** is: $\frac{x^2}{2} = y - \ln|y| + \ln 2$

SOLUTIONS OF ODE

Linear DE of the 1st order

HOMOGENOUS

$$y'(x) + p(x) \cdot y(x) = 0$$

where $p(x)$ is given function.

NON-HOMOGENOUS

$$y'(x) + p(x) \cdot y(x) = q(x)$$

where $p(x)$ and $q(x)$ are given functions.

SOLUTION OF HOMOGENOUS CASE

$$\frac{dy}{dx} = -p(x) \cdot y(x)$$

$$\frac{dy}{y(x)} = -p(x) dx \quad \text{separation}$$

$$\int \frac{dy}{y} = -\int p(x) dx \quad \text{integration}$$

$$\ln \left| \frac{y(x)}{C} \right| = -\int p(x) dx \Rightarrow$$

$$y(x) = C \exp\left(-\int p(x) dx\right) \quad \text{final solution}$$

EXAMPLE 1

$$y'(x) = y \cdot \cos x$$

boundary condition:

$$y(0) = 1$$

Solution

$$\frac{dy}{dx} = y \cos x$$

$$\frac{dy}{y} = \cos x \, dx \rightarrow \int \frac{dy}{y} = \int \cos x \, dx$$

$$\ln \left| \frac{y}{C} \right| = \sin x \Rightarrow y = C e^{\sin x}$$

The constant C is: $C = 1$

The **final solution** is: $y(x) = e^{\sin x}$

EXAMPLE 2

$$y'(x) - xy(x) = 0 \quad \text{boundary condition:} \quad y(1) = 1$$

Solution

$$\frac{dy}{dx} = yx$$

$$\frac{dy}{y} = x dx \rightarrow \int \frac{dy}{y} = \int x dx$$

$$\ln \left| \frac{y}{C} \right| = \frac{x^2}{2} \Rightarrow y = Ce^{\frac{x^2}{2}}$$

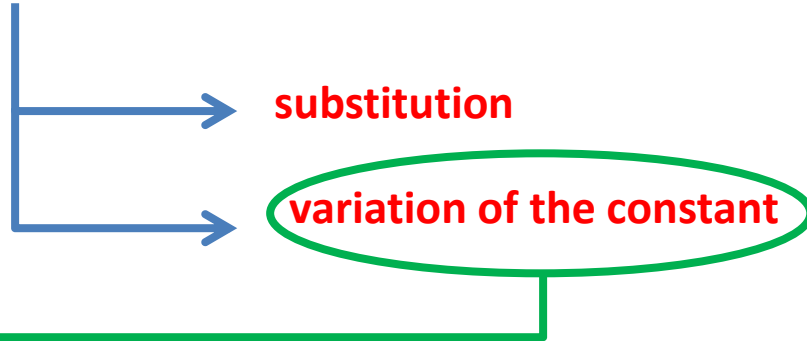
The constant C is: $C = e^{-\frac{1}{2}}$

The **final solution** is: $y(x) = y = e^{\frac{x^2}{2} - \frac{1}{2}}$

SOLUTION OF NON-HOMOGENOUS CASE

$$y'(x) + p(x) \cdot y(x) = q(x)$$

TWO METHODS



1. step: RHS is set to be 0 $y'(x) + p(x) \cdot y(x) = 0$

2. step: solving of homogenous case through separation $y(x) = C \exp\left(-\int p(x) dx\right)$

3. step: introducing constant C as the function of "x" in (2) $C \rightarrow C(x)$

4. step: differentiation of (2), where C is $C(x)$

5. step: put "y(x)" and "y'(x)" in original DE and calculate the $C(x)$

6. step: put $C(x)$ in $y(x)$ and obtain **final solution**

EXAMPLE 1

$$y'(x) + y \cdot \cos x = \sin 2x$$

Solution

1. RHS = 0 $y'(x) + y \cdot \cos x = 0$

2. separation $y = Ce^{-\sin x}$

3. constant C as the function $y = C(x)e^{-\sin x}$

4. differentiation $y' = C'e^{-\sin x} - Ce^{-\sin x} \cos x$

5. calculating of C $C'e^{-\sin x} - Ce^{-\sin x} \cos x + Ce^{-\sin x} \cdot \cos x = \sin 2x$

$$C'e^{-\sin x} = \sin 2x$$

$$C' = e^{\sin x} \sin 2x \rightarrow C(x) = \int e^{\sin x} \sin 2x \, dx$$

$$C(x) = 2e^{\sin x} (\sin x - 1) + K$$

6. final solution $y = \left(2e^{\sin x} (\sin x - 1) + K\right)e^{-\sin x} = 2(\sin x - 1) + Ke^{-\sin x}$

EXAMPLE 2

$$\cos x - y - y'(x) = 0$$

Solution

1. RHS = 0 $y' + y = 0$

2. separation $y = Ce^{-x}$

3. constant C as the function $y = C(x)e^{-x}$

4. differentiation $y' = C'e^{-x} - Ce^{-x}$

5. calculating of C $C'e^{-x} = \cos x$

$$C = \int e^x \cos x \, dx$$

$$C(x) = e^x \frac{\sin x + \cos x}{2} + K$$

6. final solution $y = \frac{\sin x + \cos x}{2} + Ke^{-x}$

SOLUTIONS OF ODE

Linear DE of the 2nd order



HOMOGENOUS

$$y''(x) + p \cdot y'(x) + q \cdot y(x) = 0$$

where p and q are real constants

NON-HOMOGENOUS

$$y''(x) + p \cdot y'(x) + q \cdot y(x) = f(x)$$

where p and q are real constants

SOLUTION OF HOMOGENOUS CASE

Solution is searched in form: $y(x) = e^{\lambda x}$

IMPORTANT
There **MUST** be **two**
linearly independent
solutions

1. step: calculation of λ $\lambda^2 + p\lambda + q = 0$ (characteristic function)

2. step: choosing of proper solution form according to discriminant D of previous equation

1. if $D = p^2 - 4q > 0 \rightarrow y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

2. if $D = p^2 - 4q < 0 \rightarrow y(x) = e^{-\frac{p}{2}x} [C_1 \cos(\omega x) + C_2 \sin(\omega x)]$

where $\omega = \sqrt{q - \frac{p^2}{4}}$

3. if $D = p^2 - 4q = 0 \rightarrow y(x) = C_1 e^{-\frac{p}{2}x} + C_2 x e^{-\frac{p}{2}x}$

SPECIAL CASE $p = 0 \wedge q = 1 \rightarrow y''(x) + y(x) = 0$

SOLUTION OF SPECIAL CASE $y(x) = C_1 \cos x + C_2 \sin x$

EXAMPLE 1

$$y''(x) - y'(x) - 2y(x) = 0, \quad y(0) = 1, y'(0) = 0$$

Solution

1. **expected solution** $y(x) = e^{\lambda x}$

2. **characteristic function** $\lambda^2 - \lambda - 2 = 0$

3. **λ** $\lambda_1 = -1, \lambda_2 = 2$

4. **general solution** $y(x) = C_1 e^{-x} + C_2 e^{2x}$

5. **boundary conditions**

$$\left. \begin{array}{l} y(0) = 1 \rightarrow C_1 + C_2 = 1 \\ y'(0) = 0 \rightarrow -C_1 + 2C_2 = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} C_1 = \frac{2}{3} \\ C_2 = \frac{1}{3} \end{array} \right.$$

6. **final solution** $y(x) = \frac{1}{3} (2e^{-x} + e^{2x})$

EXAMPLE 2

$$y''(x) - 2y'(x) + 2y(x) = 0, \quad y(0) = 1, y'(0) = 0$$

Solution

1. **expected solution** $y(x) = e^{\lambda x}$

2. **characteristic function** $\lambda^2 - 2\lambda + 2 = 0$

3. λ $D < 0 \rightarrow \lambda_{1,2} \in \mathbb{C} \quad \omega = 1$

4. **general solution** $y(x) = e^x (C_1 \cos x + C_2 \sin x)$

5. **boundary conditions**
$$\left. \begin{array}{l} y(0) = 1 \rightarrow C_1 = 1 \\ y'(0) = 0 \rightarrow C_1 + C_2 = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} C_1 = 1 \\ C_2 = -1 \end{array} \right.$$

6. **final solution** $y(x) = e^x (\cos x - \sin x)$

EXAMPLE 3

$$y''(x) + 6y'(x) + 9y(x) = 0, \quad y(0) = 1, y'(0) = 1$$

Solution

1. **expected solution** $y(x) = e^{\lambda x}$

2. **characteristic function** $\lambda^2 + 6\lambda + 9 = 0$

3. λ $D = 0 \rightarrow \lambda = -3$

4. **general solution** $y(x) = (C_1 e^{-3x} + C_2 x e^{-3x})$

5. **boundary conditions**
$$\left. \begin{array}{l} y'(0) = 1 \rightarrow -3C_1 + C_2 = 1 \\ y(0) = 1 \rightarrow C_1 = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} C_1 = 1 \\ C_2 = 4 \end{array} \right.$$

6. **final solution** $y(x) = e^{-3x} (1 + 4x)$