Mathematics for Biochemistry

LECTURE 13

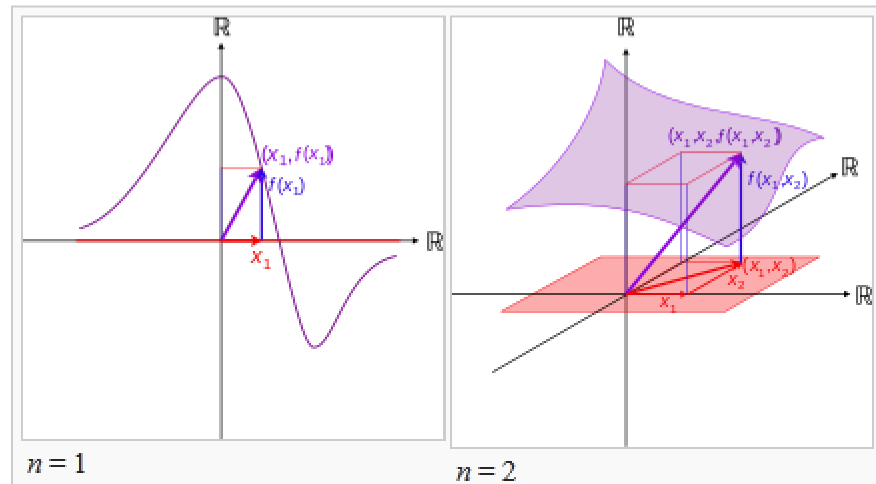
Functions of more variables 1

Content:

- basic definitions and properties
- partial and total differentiation
- differential operators

Functions of several variables:

Previously we have studied functions of one variable, y = f(x) in which x was the independent variable and y was the dependent variable. We are going to expand the idea of functions to include functions with more than one independent variable. For example, consider the functions below:



$$f(x,y) = 2x^2 + y^2$$

or

$$g(x,y,z) = 2xe^{yz}$$

or

$$h(x_1,x_2,x_3,x_4) = 2x_1 - x_2 + 4x_3 + x_4$$

In more rigorous mathematical language:

$$z : \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$z(x,y) = ax + by$$

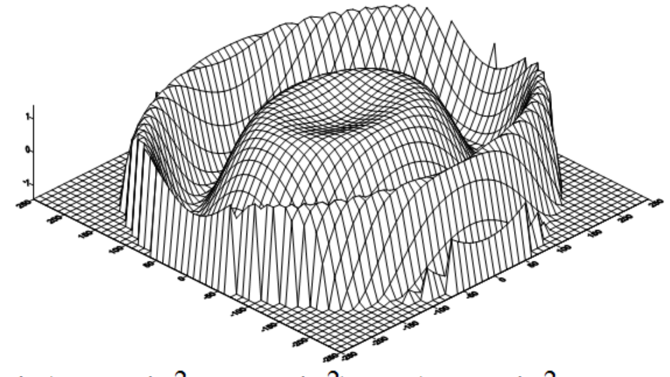
where a and b are real non-zero constants

$$z : \mathbb{R}^p \rightarrow \mathbb{R}$$
$$z(x_1, x_2, \ldots, x_p) = a_1x_1 + a_2x_2 + \cdots + a_px_p$$

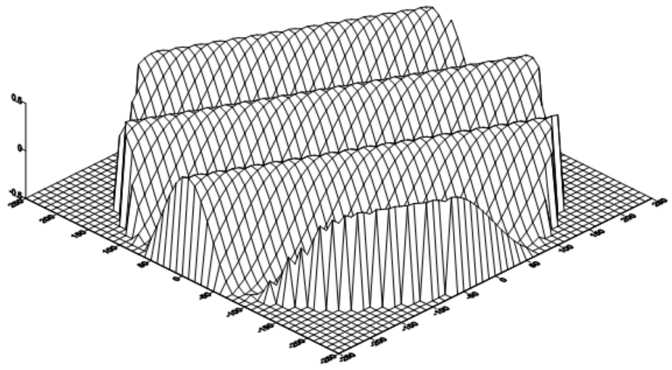
for p non-zero real constants $a_1, a_2, \ldots, a_p$

Functions of several variables:

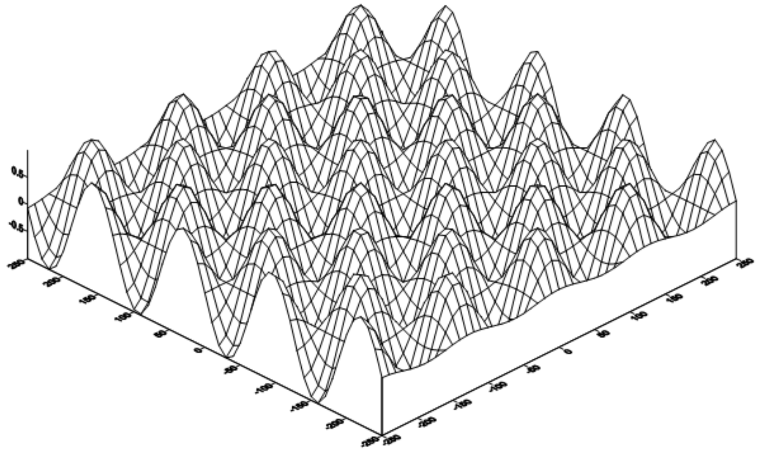
examples of graphs
for $f = f(x,y)$



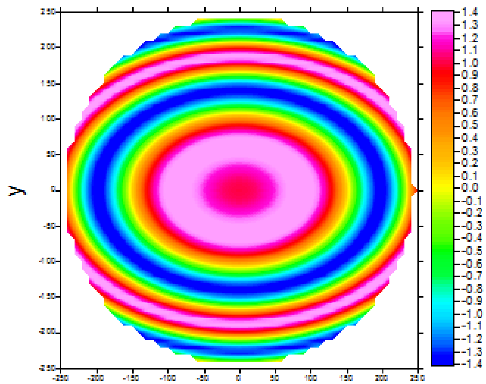
sin(0.0001*x2+0.0002*y2)+cos(0.0001*x2+0.0002*y2)

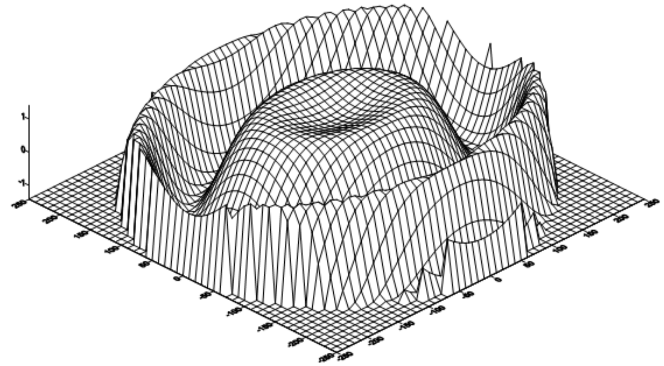


z = sin(0.01*x+0.02*y)*cos(0.01*x+0.02*y)

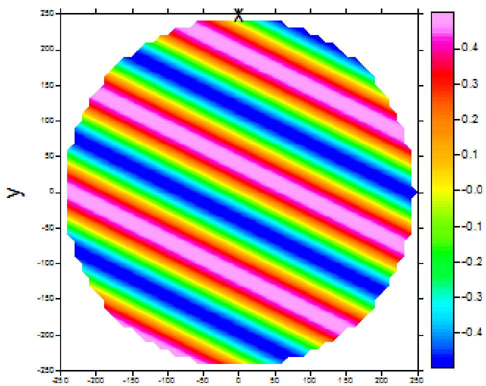


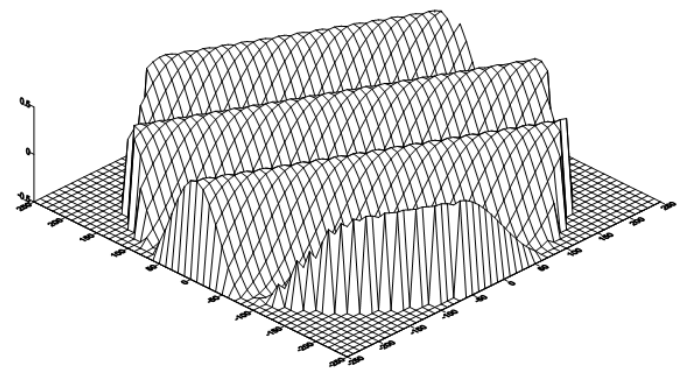
z = cos(0.05*x)*sin(0.05*y)



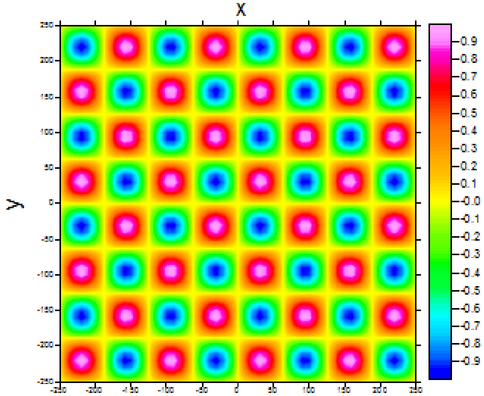


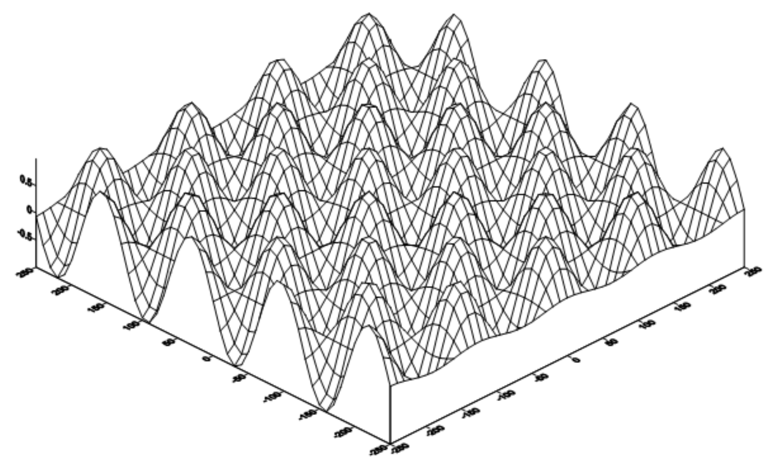
$\sin(0.0001*x^2+0.0002*y^2)+\cos(0.0001*x^2+0.0002*y^2)$





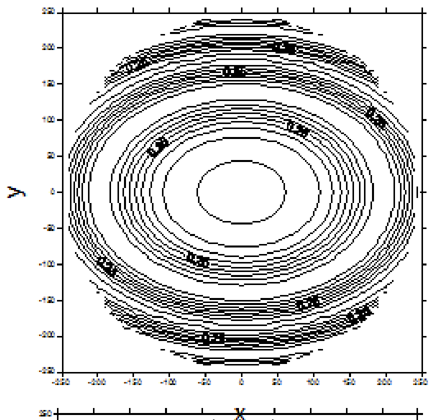
$z = \sin(0.01*x+0.02*y)*\cos(0.01*x+0.02*y)$

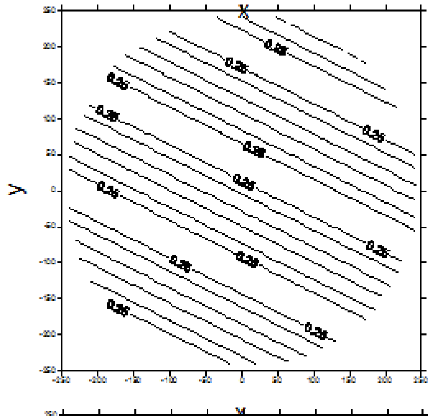


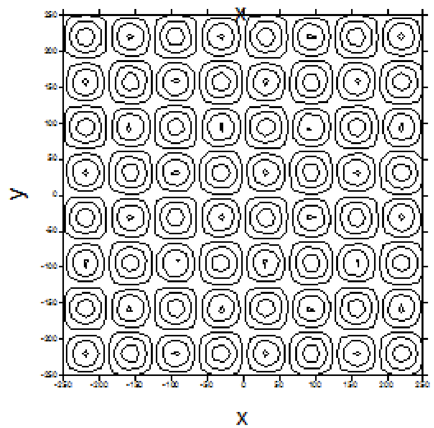


$z = \cos(0.05*x)*\sin(0.05*y)$

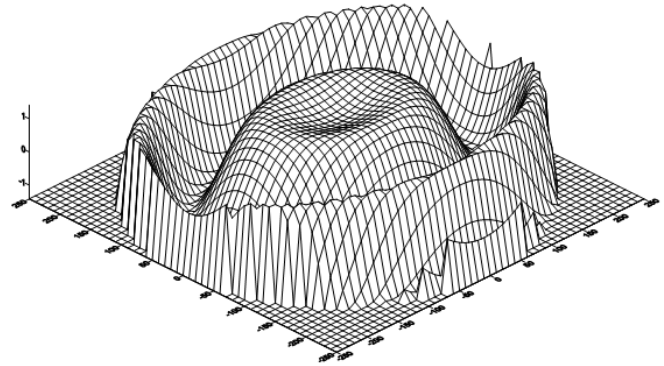
another kind of visualization
- so called <u>coloured image maps</u>
(there exist also so called contour maps)



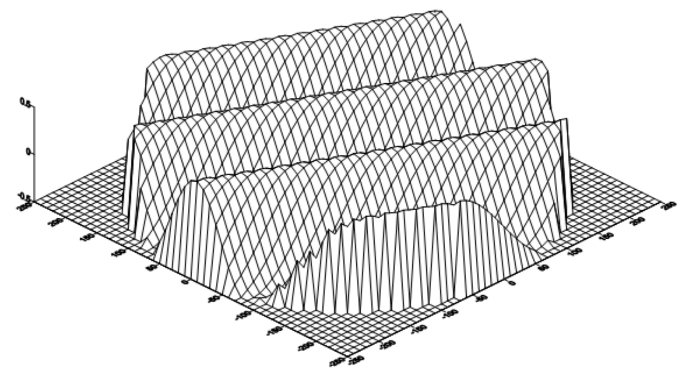




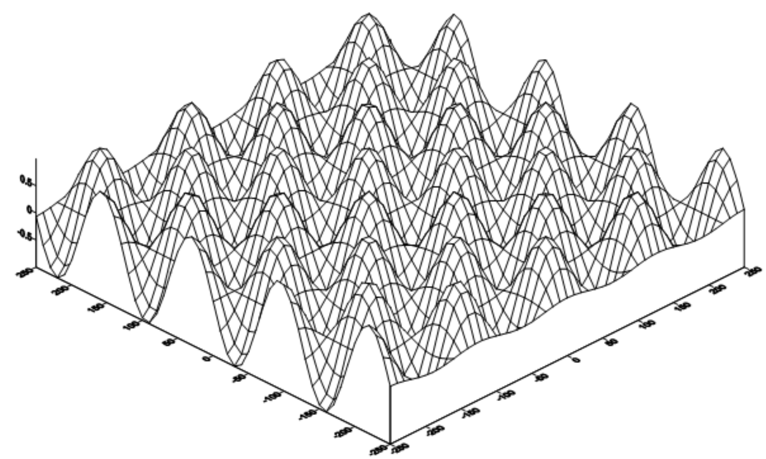
another kind of visualization
- so called coloured image maps
(there exist also so called <u>contour maps</u>)



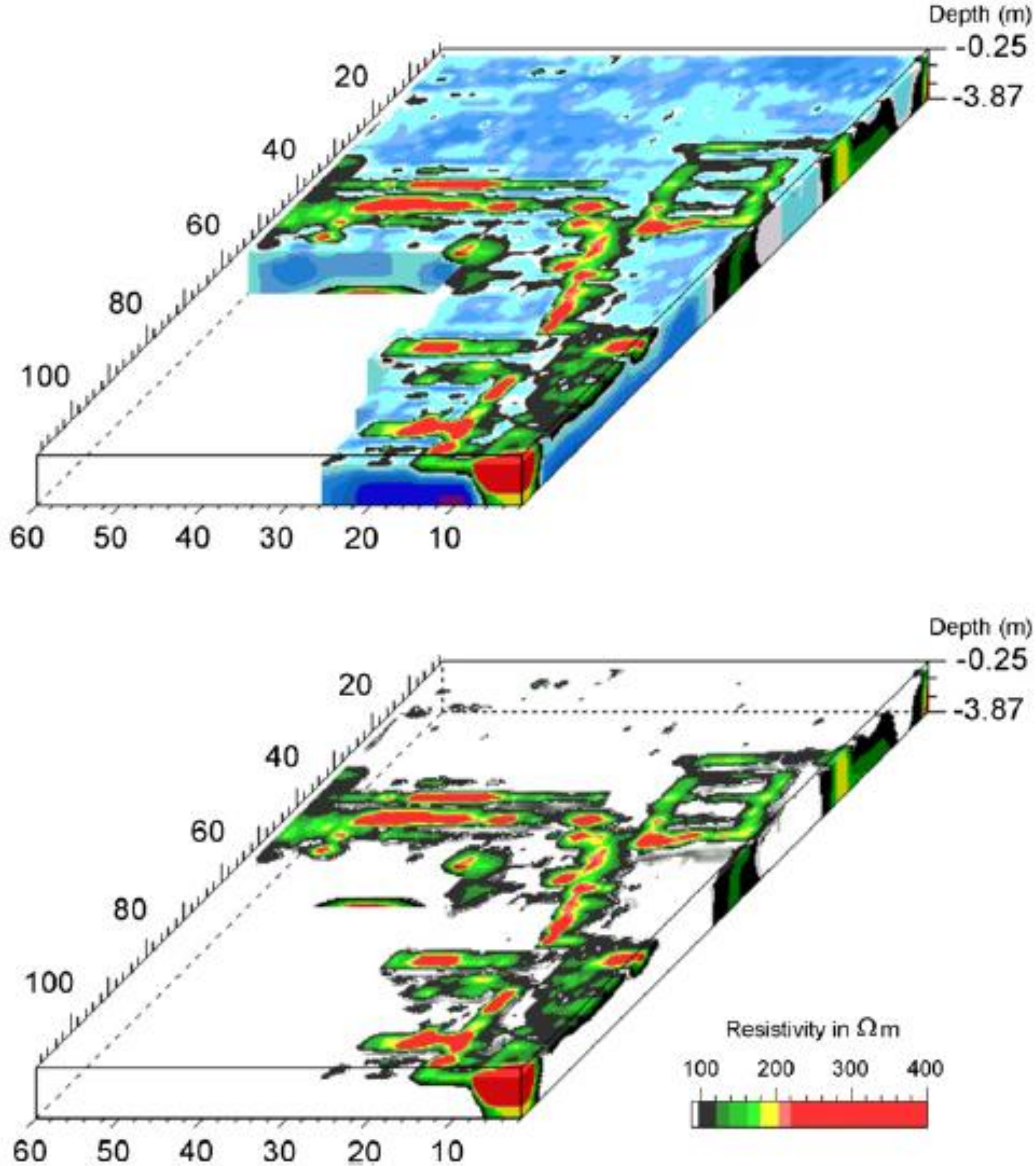
$\sin(0.0001*x^2+0.0002*y^2)+\cos(0.0001*x^2+0.0002*y^2)$



$z = \sin(0.01*x+0.02*y)*\cos(0.01*x+0.02*y)$



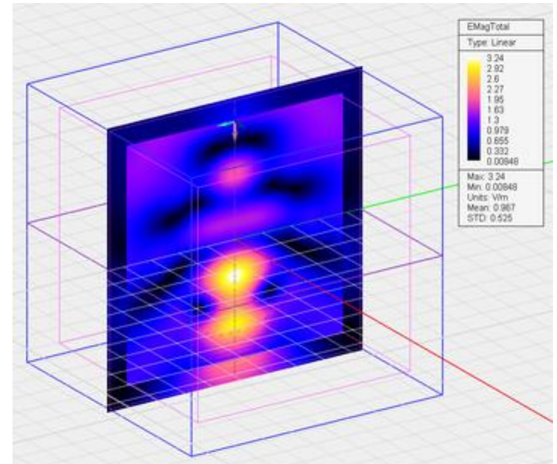
$z = \cos(0.05*x)*\sin(0.05*y)$



functions $f = f(x,y,z)$ are often visualized in form of voxel maps

Functions of several variables:

Functions of several variables are used in science for the description of various fields (physical fields, fields of properties ...).

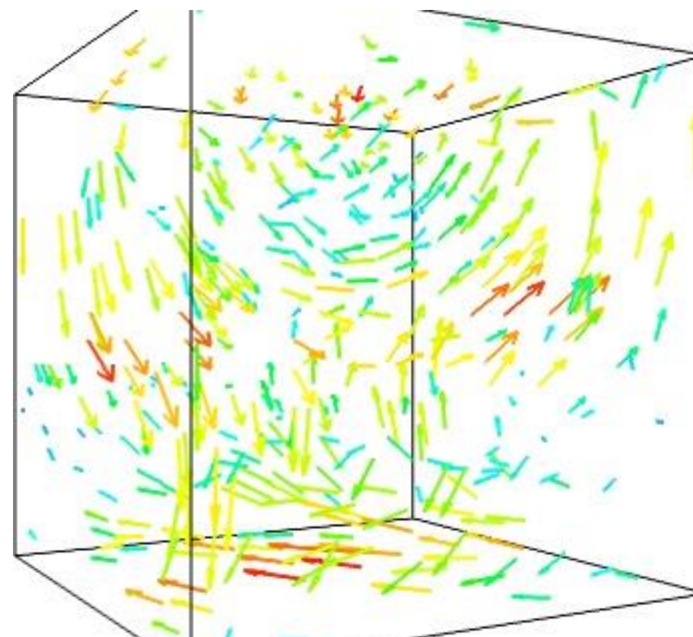


scalar fields:

e.g. temperature, density,
concentration, electric charge, ...
$t(x,y,z)$, $\rho(x,y,z)$, $U(x,y,z)$,...

and also vector fields:

e.g. electrical intensity, fluid velocity,
gravitational acceleration,...



$$\vec{A} = \mathbf{A} = \left[A_x, A_y, A_z \right]$$

$$A_x = A_x\left(x, y, z\right)$$

$$A_y = A_y\left(x, y, z\right)$$

$$A_z = A_z\left(x, y, z\right)$$

Functions of several variables:

Many properties are identical with the case of a function with one variable.

Limits and Continuity

- We say that a function $f(x,y)$ has limit L as (x,y) approaches a point (a,b) and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

  if we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b), but not equal to (a,b).

- We write also $f(x,y) \to L$ as $(x,y) \to (a,b)$ and

$$\lim_{x\to a, y\to b} f(x,y) = L$$

Functions of several variables:

Many properties are identical with the case of a function with one variable.

Continuity

- A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

- **Examples:** polynomials, rational, trigonometric, exponential, logarithmic functions are continuous on theirs domain.

With the continuity is connected also the so called distance function *d*:

$$d(\boldsymbol{x}, \boldsymbol{y}) = d(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Functions of several variables:

Some properties are new (compared with a function with one variable).

Symmetry:

A symmetric function is a function f is unchanged when two variables x_i and x_j are interchanged:

$$f(\ldots, x_i, \ldots, x_j, \ldots) = f(\ldots, x_j, \ldots, x_i, \ldots)$$

where i and j are each one of 1, 2, ..., n.

For example:

$$f(x, y, z, t) = t^2 - x^2 - y^2 - z^2$$

is symmetric in x, y, z since interchanging any pair of x, y, z leaves f unchanged, but is not symmetric in all of x, y, z, t, since interchanging t with x or y or z is a different function.

Content:

- basic definitions and properties
- partial and total differentiation
- differential operators

Functions of several variables:

Some properties are new (compared with a function with one variable).

Partial derivatives:

In the case of functions of several variables, we recognize:

a) total derivative (all variables can vary and derivatives with respect to all variables are involved)

b) partial derivative (it is a derivative with respect to one of the variables with the others held constant)

$$f'_x, \; f_x, \; \partial_x f, \; \frac{\partial}{\partial x} f, \; \text{or} \; \frac{\partial f}{\partial x}$$

Example, function $f = x^2 + xy + y^2$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + xy + y^2\right) = 2x + y + 0 = 2x + y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + xy + y^2\right) = 0 + x + 2y = x + 2y$$

another tool is given in the next slide:

Partial derivatives:

For the beginner it is helpful to imagine instead of a variable (e.g. *y*) for a moment a constant (e.g. *b*).

Example 1

Let $f(x,y) = y^3x^2$. Calculate $\frac{\partial f}{\partial x}(x,y)$.

Solution: To calculate $\frac{\partial f}{\partial x}(x,y)$, we simply view y as being a fixed number and calculate the ordinary derivative with respect to x. The first time you do this, it might be easiest to set $y = b$, where b is a constant, to remind you that you should treat y as though it were number rather than a variable. Then, the partial derivative $\frac{\partial f}{\partial x}(x,y)$ is the same as the ordinary derivative of the function $g(x) = b^3x^2$. Using the rules for ordinary differentiation, we know that

$$\frac{\mathrm{d}g}{\mathrm{d}x}(x) = 2b^3x.$$

Now, we remember that $b = y$ and substitute y back in to conclude that

$$\frac{\partial f}{\partial x}(x,y) = 2y^3x.$$

Partial derivatives – few examples:

1. If $z = f(x, y) = x^4y^3 + 8x^2y + y^4 + 5x$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = 4x^3y^3 + 16xy + 5 \qquad \text{(Note: } y \text{ fixed, } x \text{ independent variable, } z \text{ dependent variable)}$$

$$\frac{\partial z}{\partial y} = 3x^4y^2 + 8x^2 + 4y^3 \qquad \text{(Note: } x \text{ fixed, } y \text{ independent variable, } z \text{ dependent variable)}$$

2. If $z = f(x, y) = (x^2 + y^3)^{10} + \ln(x)$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = 20x(x^2 + y^3)^9 + \frac{1}{x} \qquad \text{(Note: We used the chain rule on the first term)}$$

$$\frac{\partial z}{\partial y} = 30y^2(x^2 + y^3)^9 \qquad \text{(Note: Chain rule again, and second term has no } y\text{)}$$

3. If $z = f(x, y) = xe^{xy}$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = e^{xy} + xye^{xy} \qquad \text{(Note: Product rule (and chain rule in the second term)}$$

$$\frac{\partial z}{\partial y} = x^2e^{xy} \qquad \text{(Note: No product rule, but we did need the chain rule)}$$

Functions of several variables:

Some properties are new (compared with a function with one variable).

Total derivative (differential):

In the case of functions of several variables, we recognize:

a) total derivative (all variables can vary and derivatives with respect to all variables are involved)

b) partial derivative (it is a derivative with respect to one of the variables with the others held constant)

For a function z = f(x, y, .. , u) the total differential is defined as

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \cdots + \frac{\partial z}{\partial u} du\,.$$

Example, function $f = x^2 + xy + y^2$:

$$df = \frac{\partial}{\partial x}\left(x^2 + xy + y^2\right)dx + \frac{\partial}{\partial y}\left(x^2 + xy + y^2\right)dy = \left(2x + y\right)dx + \left(2y + x\right)dy$$

Differential operators

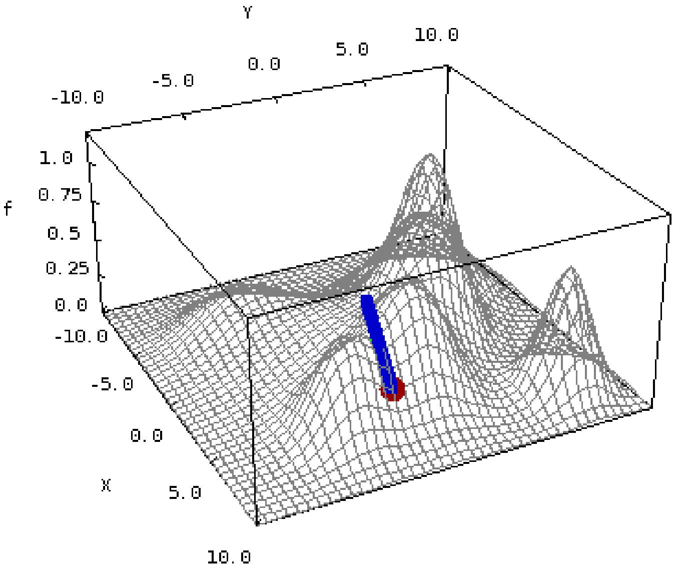
There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (div grad)

These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Gradient – show the direction and size of the greatest change of a scalar field in each point of its domain,
input of the operation: scalar field
output of the operation: vector field



$$gradU = \mathbf{A} = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are elementary vectors

Comment to the notation:
We can write gradient using the so called nabla or del operator ∇:

$$gradU = \nabla U$$

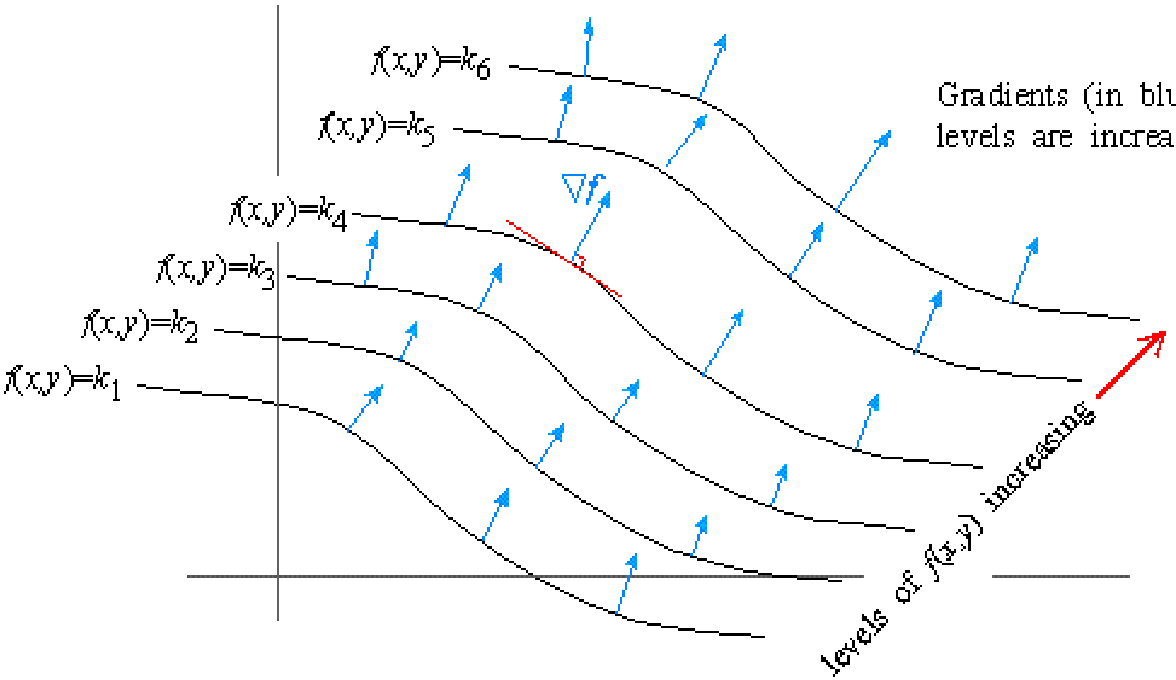
where

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Differential operators:

Gradient – show the direction and size of the greatest change of a scalar field in each point of its domain.

$$gradU = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}$$



In physical fields, gradient is always pointing in the direction of force lines (perpendicular to equipotential lines).

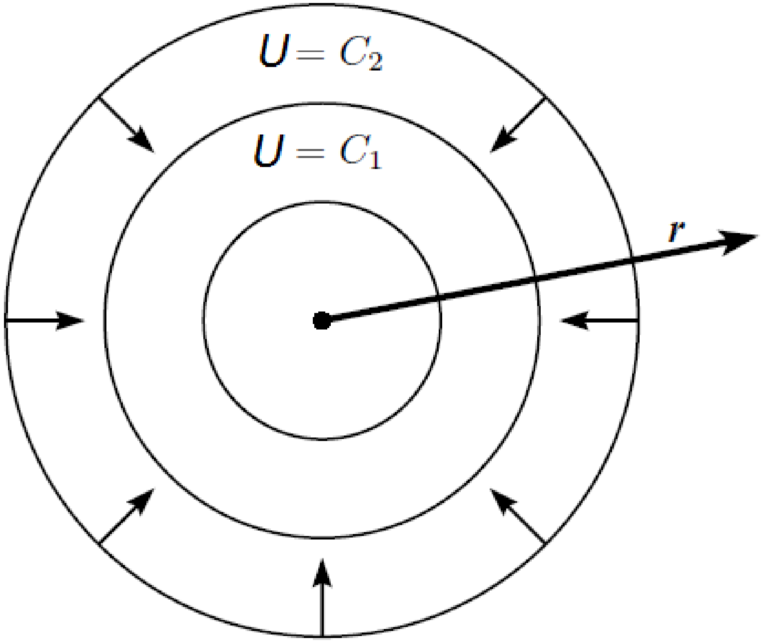
Gradient – example (field of positive electrical charge): (1/3)

Electrical potential *U*, caused by a positive electrical point charge (*Q*), situated in the origin of the coordinate system (Cartesian) can be described by means of the following equation:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2+y^2+z^2}}$$

where ε_0 is the electrical permittivity of vacuum ($8.854\cdot10^{-12}$ F/m).

Equipotential surfaces of this scalar field build spherical surfaces around the origin of the coordinate system. Gradient is a vector field, which vectors point in each point of the space perpendicular to these equipotential surfaces.



Gradient – example (field of positive electrical charge): (2/3)

$$gradU = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}$$

We will evaluate the gradient of this scalar function:

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{x^2 + y^2 + z^2}}$$

because the field of electrical intensity (vector) is given: $\vec{E} = -gradU$

First we evaluate the partial derivatives of *U* with respect to *x*, *y* and *z*.

$$\frac{\partial U}{\partial x} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left(\left[x^2 + y^2 + z^2\right]^{-\frac{1}{2}}\right) = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\left(\left[x^2 + y^2 + z^2\right]^{-\frac{3}{2}} 2x\right) =$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{\left[x^2 + y^2 + z^2\right]^{\frac{3}{2}}}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right)$$

Partial derivatives $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$ are evaluated in a very similar way.

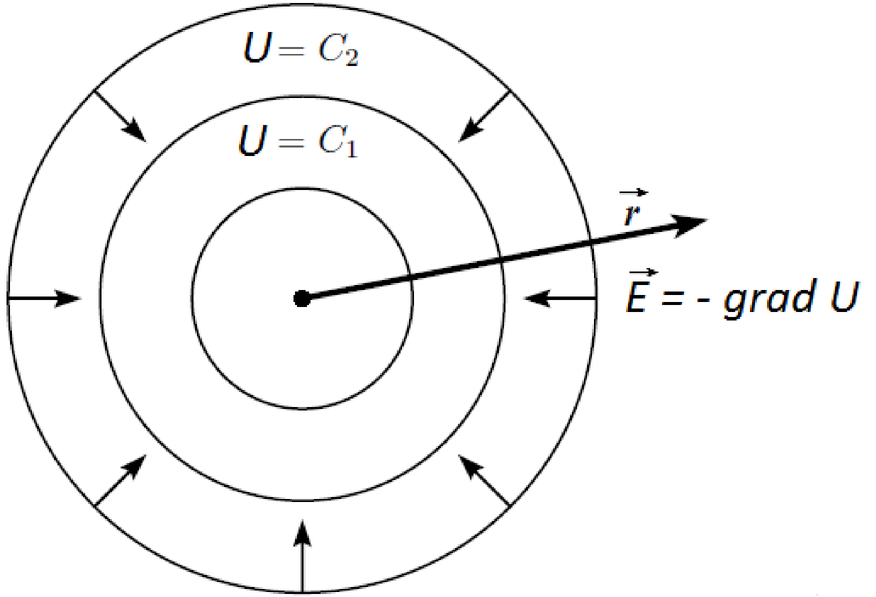
Gradient – example (field of positive electrical charge): (3/3)

$$\frac{\partial U}{\partial x} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right), \qquad \frac{\partial U}{\partial y} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{r^3}\right), \qquad \frac{\partial U}{\partial z} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

$$\mathbf{E} = -gradU = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\mathbf{i} + \frac{y}{r^3}\mathbf{j} + \frac{z}{r^3}\mathbf{k}\right) = \boxed{\frac{Q}{4\pi\varepsilon_0}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^3}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}}$$

This is a vector field, pointing in the same direction as the vector $\vec{r}$ and having the size:

$$|\mathbf{E}| = \frac{Q}{4\pi\varepsilon_0}\frac{r}{r^3} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$$



Differential operators:

There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (divgrad)

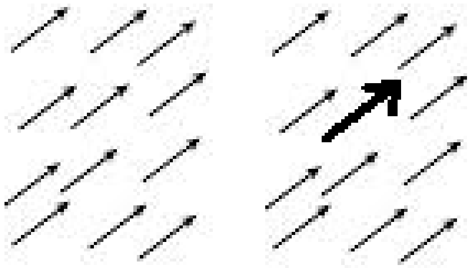
These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Divergence – tells about the sources of a vector field: when the result is zero then there is no source of the field in the point.

input of the operation: components of vector field

output of the operation: scalar value field



$$div\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

where A_x, A_y, A_z are the components of vector $\mathbf{A}$.

Comment: Divergence depends on the changes of the size of vector components and not the change of their direction.

Comment to the notation:

We can write also divergence using the nabla or del operator ∇:

$$div\mathbf{A} = \nabla \cdot \mathbf{A}$$

where

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Divergence – example (field of electrical charge): (1/2)

Field of electrical intensity (a vector field) is given by:

$$\mathbf{E} = -gradU = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^3}\right) = E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}$$

$$E_x = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right),\ \mathrm{E_y} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{r^3}\right),\ \mathrm{E_z} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

To evaluate the divergence of this field, we need to evaluate the following derivatives:

$$\frac{\partial \mathrm{E_x}}{\partial \mathrm{x}} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{x}}\left(\frac{x}{r^3}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{x}}\left(\frac{x}{\left[x^2+y^2+z^2\right]^{3/2}}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{x}}\left(x\left[x^2+y^2+z^2\right]^{-3/2}\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} + x\left(\frac{-3}{2}\right)\left[x^2+y^2+z^2\right]^{-5/2}2x\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3x^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

Divergence – example (field of electrical charge): (2/2)

For all three derivatives we get:

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3x^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3y^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-3/2} - 3z^2\left[x^2+y^2+z^2\right]^{-5/2}\right)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left(3\left[x^2+y^2+z^2\right]^{-3/2} - 3\left(x^2+y^2+z^2\right)\left[x^2+y^2+z^2\right]^{-5/2}\right) =$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(3\left[x^2+y^2+z^2\right]^{-3/2} - 3\left[x^2+y^2+z^2\right]^{-3/2}\right) \boxed{= 0}$$

This result is valid for all points with the exception of the coordinate system origin, where x = y = z = 0 (source area).

Differential operators:

There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (divgrad)

These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Rotation – tells about the existence of so called curls of the vector field (not about the sources).

input of the operation: components of vector field

output of the operation: vector field

$$rot\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Comment: Rotation does not depend on the changes of the size of vector components (this was the role of divergence).

Comment to the notation:

We can write also divergence using the nabla or del operator ∇:

$$rot\vec{A} = \nabla \times \mathbf{A}$$

where

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Rotation – example (field of electrical charge): (1/2)

Field of electrical intensity (a vector field) is given by:

$$\mathbf{E} = -gradU = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^3}\right) = E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k}$$

$$E_x = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\right),\ \mathrm{E_y} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{y}{r^3}\right),\ \mathrm{E_z} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{z}{r^3}\right)$$

For the rotation evaluation we need following derivatives:

$$\frac{\partial \mathrm{E_z}}{\partial \mathrm{y}} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{y}}\left(\frac{z}{r^3}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{y}}\left(\frac{z}{\left[x^2+y^2+z^2\right]^{\frac{3}{2}}}\right) = \frac{zQ}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{y}}\left(\left[x^2+y^2+z^2\right]^{-\frac{3}{2}}\right) =$$

$$= \frac{zQ}{4\pi\varepsilon_0}\left(\left(\frac{-3}{2}\right)\left[x^2+y^2+z^2\right]^{-\frac{5}{2}} 2y\right) = \frac{-3yzQ}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-\frac{5}{2}}\right)$$

$$\frac{\partial \mathrm{E_y}}{\partial \mathrm{z}} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial \mathrm{z}}\left(\frac{y}{\left[x^2+y^2+z^2\right]^{\frac{3}{2}}}\right) = \frac{-3zyQ}{4\pi\varepsilon_0}\left(\left[x^2+y^2+z^2\right]^{-\frac{5}{2}}\right)$$

Rotation – example (field of electrical charge): (2/2)

From the evaluated derivatives it follows:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

In a similar way we can show:

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

... and for the rotation it is valid:

$$rot\mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = \mathbf{0}$$

This result is valid for all points with the exception of the coordinate system origin, where x = y = z = 0 (source area).

Differential operators:

There exist few special operations, which use partial derivatives and express properties of analyzed functions of several variables – so called **differential operators**:

- gradient (grad)
- divergence (div)
- rotation (rot)
- Laplacian operator (divgrad)

These are used in various descriptions and derivations of basic properties of physical fields.

Differential operators:

Laplacian operator – in mathematical physics is often used the following (combined) differential operator,
input of the operation: scalar field
output of the operation: scalar field

$$div(gradU) = \frac{\partial(\partial U/\partial x)}{\partial x} + \frac{\partial(\partial U/\partial y)}{\partial y} + \frac{\partial(\partial U/\partial z)}{\partial z}$$

$$div(gradU) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Comment to the notation:
We can write gradient using the so called nabla or del operator ∇:

$$div(gradU) = \nabla \cdot (\nabla U) = \nabla^2 U = \Delta U$$

Differential operators:

Beside this combined operator (Laplacian), the are valid following equations:

$$rot\left(grad\,U\right) \equiv 0$$

$$div\left(rot\,\mathbf{A}\right) \equiv 0$$

These equations have important impacts on the properties of some physical fields:

- the first one tells that so called potential fields (which intensity can be expressed by means of the gradient) can not build curls,
- the second one tells us that in a curl there are no sources.

You can try to check it mathematically (make a proof) in a frame of a homework.