Mathematics for Biochemistry

LECTURE 15

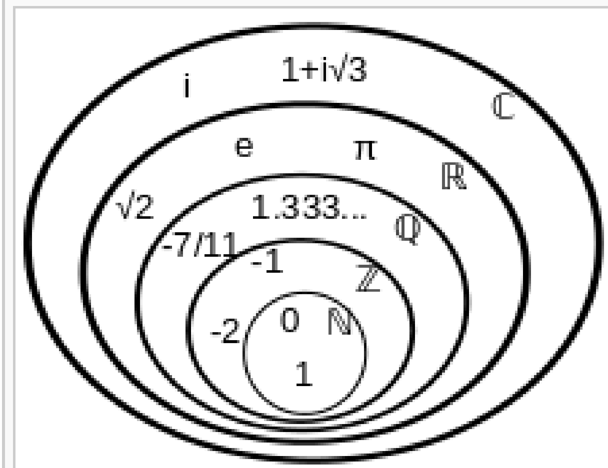
Complex numbers

Content:

- complex numbers, introduction
- complex numbers, basic operations
- complex numbers, functions

Basic objects in mathematics: **numbers, sets**

Different types of numbers have many different uses.
Numbers can be classified into sets, called **number systems**.



Subsets of the complex numbers.

Main number systems

$\mathbb{N}$	**Natural**	0, 1, 2, 3, 4, ... **or** 1, 2, 3, 4, ...
$\mathbb{Z}$	**Integer**	..., −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, ...
$\mathbb{Q}$	**Rational**	$\frac{a}{b}$ where *a* and *b* are integers and *b* is not 0
$\mathbb{R}$	**Real**	The limit of a convergent sequence of rational numbers
$\mathbb{C}$	**Complex**	*a* + *bi* where *a* and *b* are real numbers and *i* is the square root of −1

$1.5 = \frac{3}{2}$ Ratio

Rational

$\pi = 3.14159... = \frac{?}{?}$ (No Ratio)

Irrational

i – so called imaginary unit

Complex numbers - introduction

Definition: A complex number is a number that can be expressed in the form $a + ib$, where a and b are real numbers and i is the imaginary unit, that satisfies the equation $i^2 = -1$.

a – the so-called real part of the complex number,

b – the so-called imaginary part of the complex number.

Complex numbers extend the concept of the one-dimensional number line to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part.

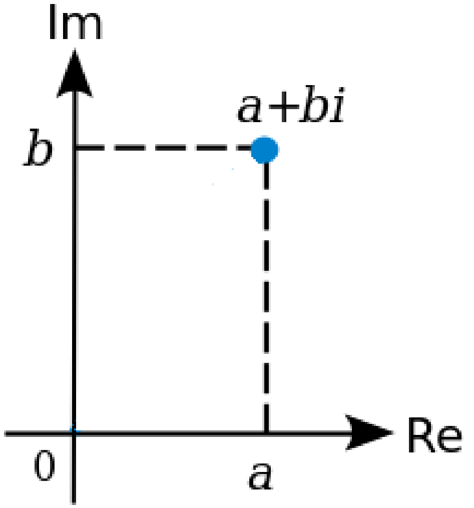
The complex number $a + ib$ can be identified with the point (a, b) in the complex plane.

A complex number whose real part is zero is said to be purely imaginary ($0+ib$), whereas a complex number whose imaginary part is zero is a real number ($a+i0$). In this way, the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone.

Example:

$$\mathrm{Re}(-3.5 + 2i) = -3.5$$

$$\mathrm{Im}(-3.5 + 2i) = 2$$



Complex numbers - introduction

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Imaginary unit *i* was not selected randomly as $i^2 = -1$, but because this plays very important role in various solutions of equations and other mathematical problems.

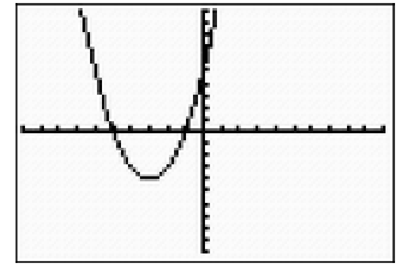
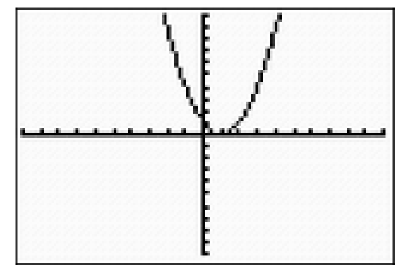
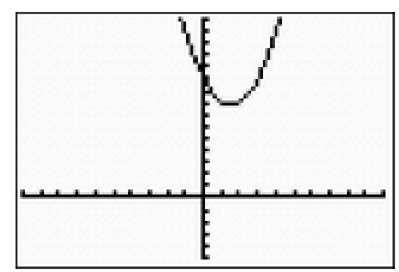
Example (quadratic equation with negative discriminant):

$$x^2 - 3x + 10 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 10}}{2} = \frac{3 \pm \sqrt{-31}}{2} = \frac{3 \pm i\sqrt{31}}{2} = \begin{matrix} x_1 = 3/2 + (i/2)\sqrt{31} \\ x_2 = 3/2 - (i/2)\sqrt{31} \end{matrix}$$

geometrical presentation of quadratic equation solutions:

Value of the discriminant	Example showing nature of roots of $ax^2 + bx + c = 0$	Graph indicating x-intercepts $y = ax^2 + bx + c$
POSITIVE $b^2 - 4ac > 0$	$x^2 + 6x + 5 = 0$ $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$ $x = \frac{-6 \pm \sqrt{16}}{2} = \frac{-6 \pm 4}{2}$ $x = -1; \quad x = -5$ **There are two real roots.** (If the discriminant is a perfect square, the two roots are rational numbers. If the discriminant is not a perfect square, the two roots are irrational numbers containing a radical.)	 **There are two x-intercepts.**
ZERO $b^2 - 4ac = 0$	$x^2 - 2x + 1 = 0$ $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$ $x = \frac{2 \pm \sqrt{0}}{2} = 1$ $x = 1; \quad x = 1$ **There is one real root.** (The root is repeated.)	 **There is one x-intercept.**
NEGATIVE $b^2 - 4ac < 0$	$x^2 - 3x + 10 = 0$ $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(10)}}{2(1)}$ $x = \frac{3 \pm \sqrt{-31}}{2}$ $x = \frac{3 + i\sqrt{31}}{2}; \quad x = \frac{3 - i\sqrt{31}}{2}$ **There are two complex roots.**	 **There are no x-intercepts.**

Complex numbers - introduction

Definition: A complex number is a number that can be expressed in the form *a* + *ib*, where *a* and *b* are real numbers and *i* is the imaginary unit, that satisfies the equation $i^2 = -1$.

a – the so-called real part of the complex number,

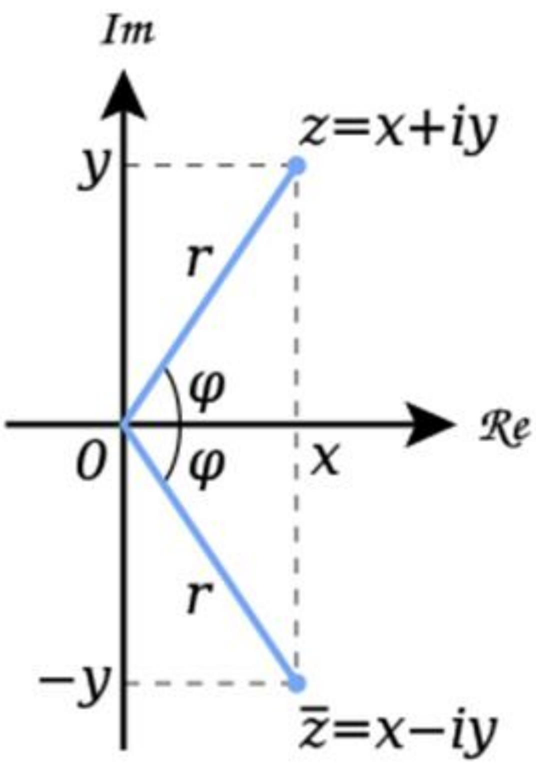
b – the so-called imaginary part of the complex number.

back to the previous example:

$$x_1 = 3/2 + (i/2)\sqrt{31}$$
$$x_2 = 3/2 - (i/2)\sqrt{31}$$

Complex solutions are plotted in the complex plane (so called Argand diagram):

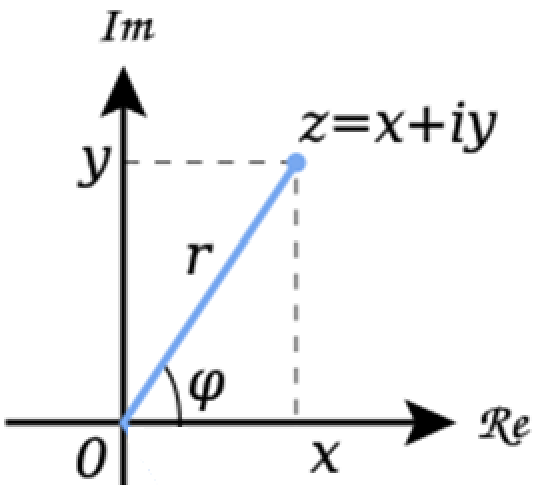
$z = a + ib = x + iy$



Comment: $\bar{z}$ is called as complex conjugate number (imaginary part has opposite sign).

Complex numbers - introduction

Form: A complex number can be written in two basic forms:



1. Cartesian form (with real and imaginary parts):

   $z = a + ib = x + iy$

2. goniometric or exponential form

   (using the so-called Euler's formula):

   $z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}.$

where:

r – modulus (magnitude) of a complex number

(also called as absolute value: $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$)

φ – argument of the complex number: $Arg(z) = \varphi = arctg(y/x)$

it is also valid (for k = 0, 1, …): $\arg(z) = \varphi = arctg(y/x) + 2\pi k$

Comment to the imaginary unit:

$i = \sqrt{-1}$, $i^1 = i$, $i^2 = -1$, $i^3 = i \cdot i^2 = -i$, $i^4 = i^2 \cdot i^2 = 1$, $i^5 = i^4 \cdot i = i$

$i^{-1} = i/i^2 = -i$, …

Complex numbers – introduction

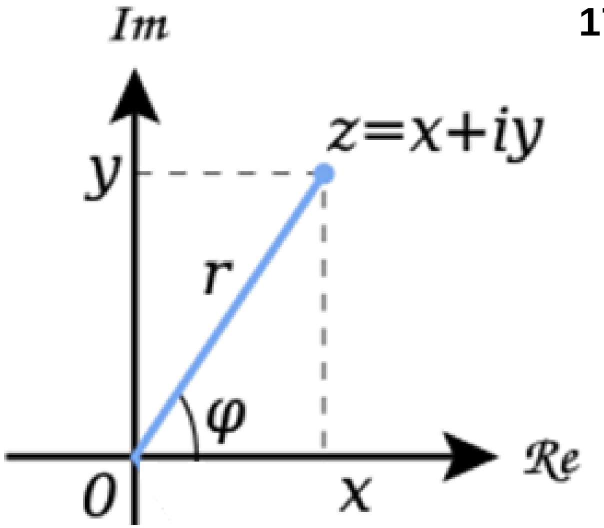
Leonhard Euler
1707 - 1783

Euler's formulas:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi\,,\quad e^{-i\varphi} = \cos\varphi - i\sin\varphi,$$

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$



Derivation (proof) of these formulas via Taylor series.

Complex numbers – basic operations

Equality:

Two complex numbers are equal if and only if both their real and imaginary parts are equal.

We can write this in math-symbols:

$$z_1 = z_2 \leftrightarrow (\mathrm{Re}(z_1) = \mathrm{Re}(z_2) \wedge \mathrm{Im}(z_1) = \mathrm{Im}(z_2))$$

Conjugation:

The complex conjugate of the complex number $z = x + iy$ is defined to be $x - iy$. It is denoted $\bar{z}$ or z^* .

Geometrically, conjugate complex number is the "reflection" of the complex number about the real axis. Conjugating twice gives the original complex number:

Conjugation distributes over the standard arithmetic operations:

$$\overline{z+w} = \bar{z} + \bar{w},$$
$$\overline{z-w} = \bar{z} - \bar{w},$$
$$\overline{zw} = \bar{z}\bar{w},$$
$$\overline{(z/w)} = \bar{z}/\bar{w}.$$

Complex numbers – basic operations

Addition and subtraction:

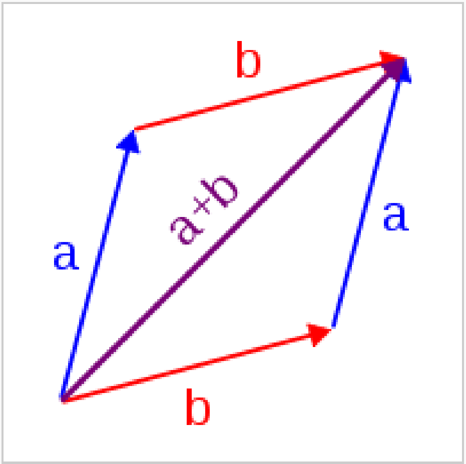
Complex numbers are added by adding the real and imaginary parts of the summands.

Addition:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

Similarly, subtraction:

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$



Addition of two complex numbers can be done geometrically by constructing a parallelogram.

Multiplication and division:

The multiplication of two complex numbers is defined by the following formula:

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

And, division:

$$\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

Complex numbers – basic operations

Multiplication and division:

We can use in a much more straight-forward way the exponential form of complex numbers:

$$z_1 z_2 = r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1+\varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}$$

Or in the goniometric form:

$$z_1 z_2 = r_1 r_2 \left[\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)\right]$$

Complex numbers – basic operations

Raising to a power (exponentiation):

Here we can express this kind of operation much more effectively again with the exponential form of complex numbers:

$$z^n = \left[r e^{i\phi} \right]^n = r^n e^{in\phi}$$

In Cartesian form this operation is more complicated and it utilises the so called binomial rule:

$$z^n = (a + ib)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (ib)^k$$

where $$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Complex numbers – basic operations

n^{th} root evaluation:

Also here we can express this kind of operation much more effectively again with the exponential form of complex numbers:

$$\sqrt[n]{z} = z^{\frac{1}{n}} = \left[re^{i\phi} \right]^{\frac{1}{n}} = \left[re^{i(\phi + 2k\pi)} \right]^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i(\varphi + 2k\pi)}{n}} =$$

$$= r^{\frac{1}{n}} e^{i\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)} = \sqrt[n]{r}\left[\cos\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)\right]$$

Here we can see that the result of nth root of a complex number is not unique – it has n solutions (for $k = 0, 1, 2, \ldots n-1$).

These solutions lie on a circle with the radius $r^{1/n}$ and their arguments are different by $2\pi/n$.

Example

$$Z = 1 + i \qquad \to \sqrt[4]{Z} = ?$$

$$a = 1, \quad b = 1 \quad \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\varphi = \arctan\frac{1}{1} = \frac{\pi}{4}$$

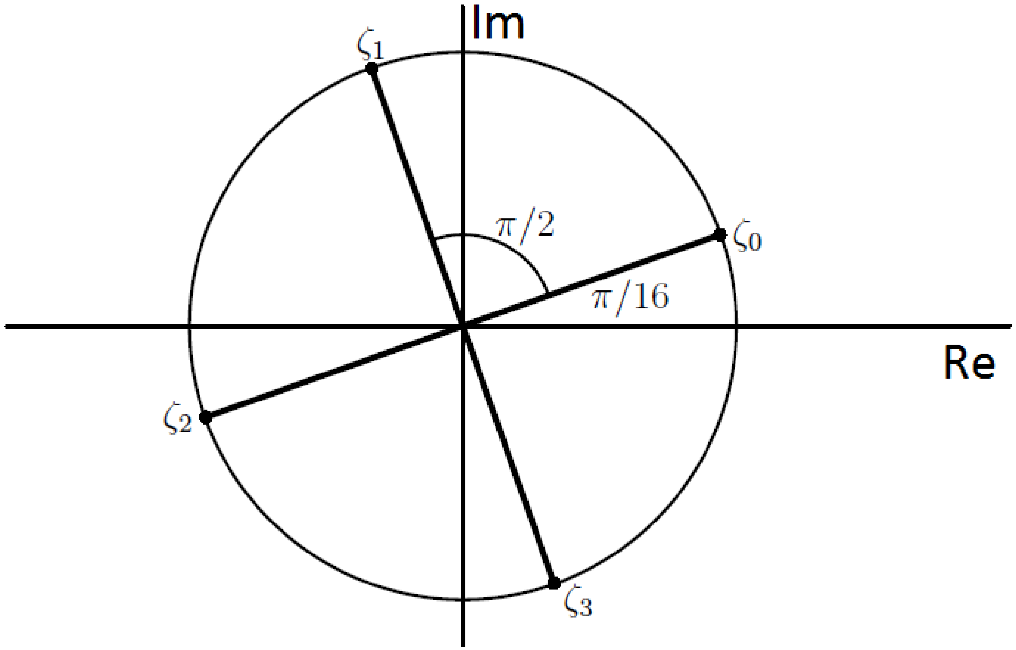
$$\Rightarrow Z = \sqrt{2}e^{i\frac{\pi}{4}} \to n = 4 \to \sqrt[4]{Z} = Z^{\frac{1}{4}} = \left[\sqrt{2}e^{i\left(\frac{\pi}{4}+2k\pi\right)}\right]^{\frac{1}{4}} = 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{2k\pi}{4}\right)} = 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)}$$

$$k = 0 \quad \to 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)} = 2^{\frac{1}{8}}e^{i\frac{\pi}{16}}$$

$$k = 1 \quad \to 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{\pi}{2}\right)} = 2^{\frac{1}{8}}e^{i\frac{9}{16}\pi}$$

$$k = 2 \quad \to 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{2\pi}{2}\right)} = 2^{\frac{1}{8}}e^{i\frac{17}{16}\pi}$$

$$k = 3 \quad \to 2^{\frac{1}{8}}e^{i\left(\frac{\pi}{16}+\frac{3\pi}{2}\right)} = 2^{\frac{1}{8}}e^{i\frac{25}{16}\pi}$$



Complex numbers – functions

We distinguish here two basic types:

a) complex functions of real variable,

b) complex functions of complex variable.

A) Complex function of real variable:

we call $z = \mathrm{f}(t)$ as complex function of real variable, when $z \in \mathrm{C}$ and $t \in \mathrm{R}$.

Examples:

1. $z = t\,\mathrm{e}^{i\alpha}$, $t \in <0, +\infty>$, $\alpha = \mathrm{const.}$

Values of t from the interval $<0, +\infty>$ are displayed into a half-line in the complex plane (forming an angle α with the real axis).

$z = t\,\mathrm{e}^{i\alpha} = t(\cos\alpha + i\sin\alpha) \Rightarrow x = t\cos\alpha,\ y = t\sin\alpha \Rightarrow y = x\,\mathrm{tg}\,\alpha.$

(this is an equation for a line)

A) Complex function of real variable:

we call $z = f(t)$ as complex function of real variable, when $z \in C$ and $t \in R$

Examples:

2. $z = r\,e^{it}$, r = const., $t \in \langle\alpha, \beta\rangle$, $\alpha \geq 0$, $\beta \leq 2\pi$.

Values of t from the interval $\langle\alpha, \beta\rangle$ are displayed into a circle-arc in the complex plane (forming an angle α with the real axis).

$z = r\,e^{it} = r(\cos t + i\sin t) \Rightarrow x = r\cos t,\ y = r\sin t$.

When we build sum of squares $x^2 + y^2$, we obtain:

$x^2+y^2 = r^2\cos^2 t+r^2\sin^2 t = r^2 (\cos^2 t+\sin^2 t) = r^2 \Rightarrow x^2+y^2 = r^2$,

(this is the equation of a circle)

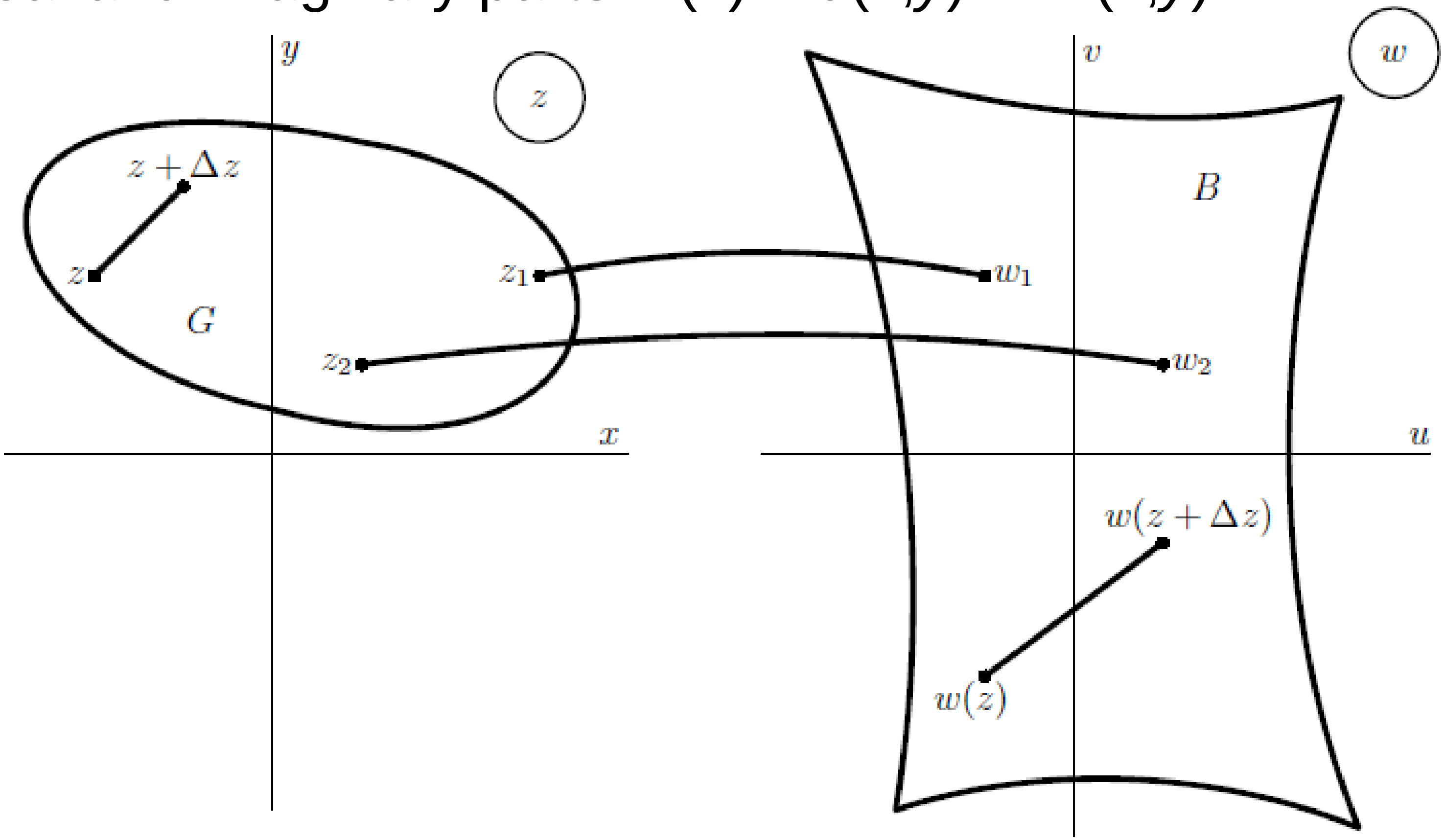
3. $z - z_0 = r\,e^{it}$, r = const., $t \in \langle 0, 2\pi\rangle$.

This is a very similar example, compared with the previous one, result is a circle in the complex plane with the centre in point z_0.

Complex numbers – functions

B) Complex function of complex variable:

we call $w = f(z)$ as complex function of complex variable, when $w \in C$ and also $z \in C$. Such a function can be divided into its real and imaginary parts: $f(z) = u(x,y) + i\,v(x,y)$.



These functions can have completely different properties like real number functions.

Complex numbers – functions

B) Complex function of complex variable:

Example: f(z) = cos(z) .

For its analysis we use one of Euler's formulas: $\cos\varphi = \dfrac{e^{i\varphi} + e^{-i\varphi}}{2}$

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

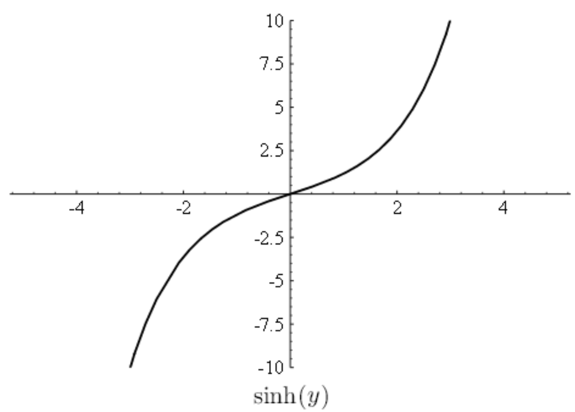
setting for $z = x + iy$ and by reformulating we get:

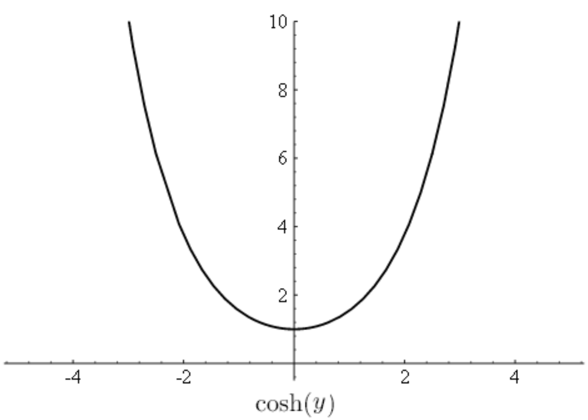
$$\cos(z) = \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

where we have used the hyperbolic functions sinh(x) and cosh(x):

$$\cosh(y) = \frac{1}{2}(e^{y} + e^{-y})$$

$$\sinh(y) = \frac{1}{2}(e^{y} - e^{-y})$$





B) Complex function of complex variable:

Example: f(*z*) = cos(*z*)

$$\cos(z) = \cos(x + \mathrm{i}\, y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

Recognising here the form of this kind of function: f(*z*) = *u*(*x*,*y*) + *i* *v*(*x*,*y*):

$$\mathrm{Re}[\cos(z)] = u(x, y) = \cos(x)\cosh(y)$$

$$\mathrm{Im}[\cos(z)] = v(x, y) = -\sin(x)\sinh(y)$$

First important thing, which we can see is that this function is not limited (like it was valid for the real function cos(*x*)), but it can grow to ±∞ due to the properties of hyperbolic sine and cosine:

$$|\cos(z)| = \left(u^2 + v^2\right)^{1/2} = \left\{[\cos(x)\cosh(y)]^2 + [\sin(x)\sinh(y)]^2\right\}^{1/2}$$

Next property is its periodicity:

cos(*z*) = cos(z + 2*k*π), when *k* = 1, 2, ...

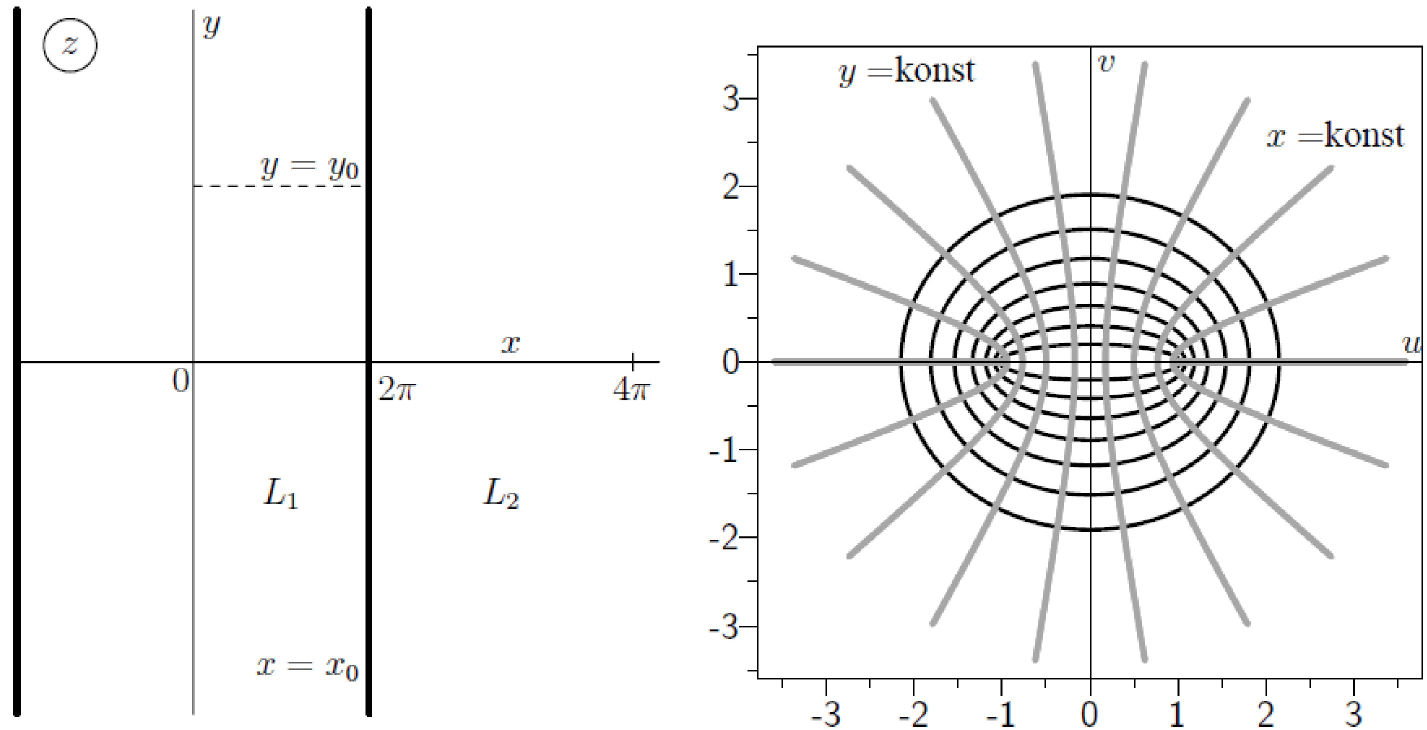
B) Complex function of complex variable:

Example: f(z) = cos(z)

$$\cos(z) = \cos(x + \mathrm{i}\, y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

Next thing is the analysis of its geometrical properties:

1. First, we take a line segment $y = y_0$, $x \in \langle 0, 2\pi \rangle$



Using $\cos^2 x + \sin^2 x = 1$ we get:

$$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1$$

This is equation of an ellipse.

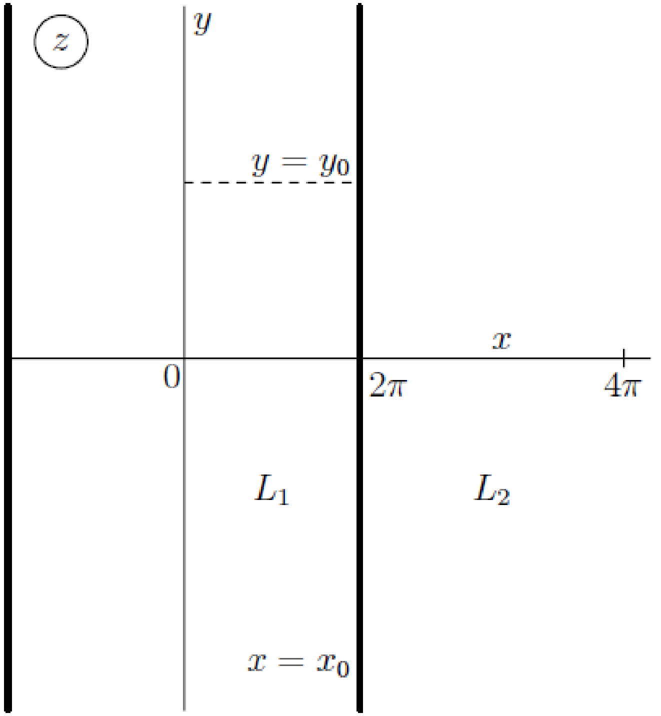
B) Complex function of complex variable:

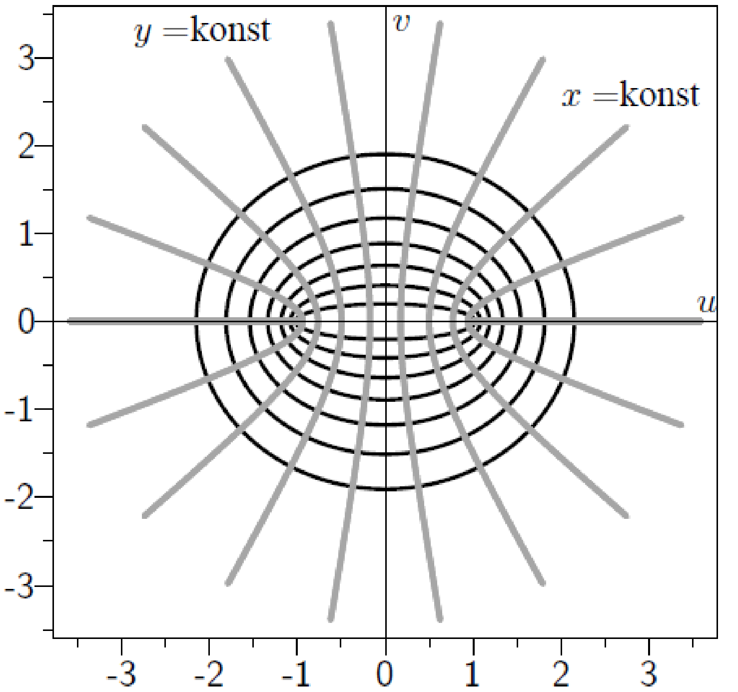
Example: f(*z*) = cos(*z*)

$$\cos(z) = \cos(x + \mathrm{i}\, y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$$

Next thing is the analysis of its geometrical properties:

2. Second, we take a line $x = x_0$, $y \in <-\infty, +\infty>$

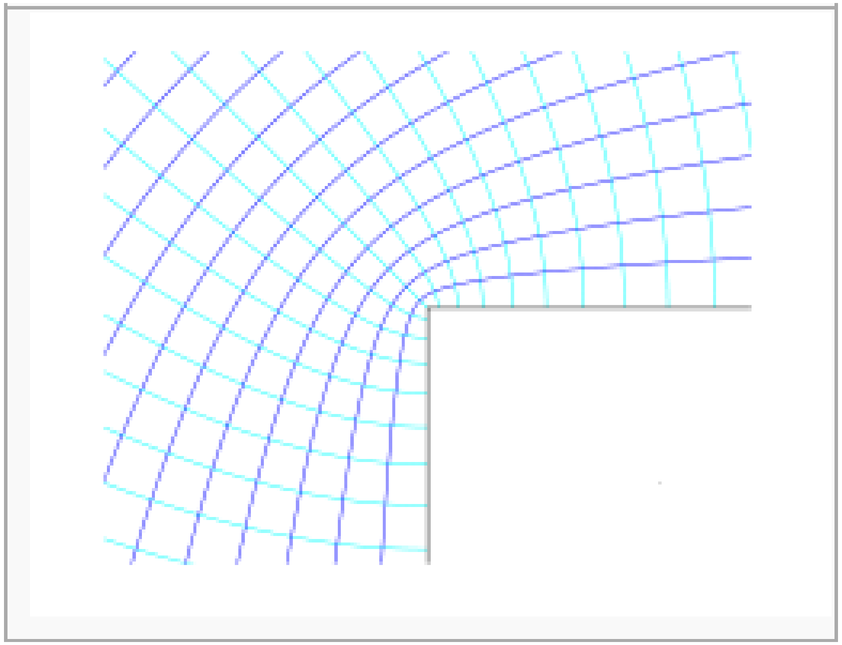




Using $\cosh^2 y - \sinh^2 y = 1$ we get:

$$\frac{u^2}{\cos^2 x_0} - \frac{v^2}{\sin^2 x_0} = 1$$

This is equation of a hyperbole.



model of physical field in close vicinity of an object
(as a result of a solution, involving complex numbers)

Applications, involving complex numbers: improper integrals, geometry, signal analysis, fluid dynamics, dynamic equations, electromagnetism, quantum mechanics, relativity