Mathematics for Biochemistry

LECTURE 16

Vectors

Content:

- Definition
- Operations

Vectors:

In mathematical physics for the description of physical quantities we recognise the following sequence of objects: **scalars**, **vectors** and **tensors**.

scalars (have only a magnitude or size)

(time, temperature, angle, length, ...) t

vectors (have magnitude and direction)

(strength, velocity, acceleration, ...) $\mathbf{a}, \mathbf{A}, \vec{a}$

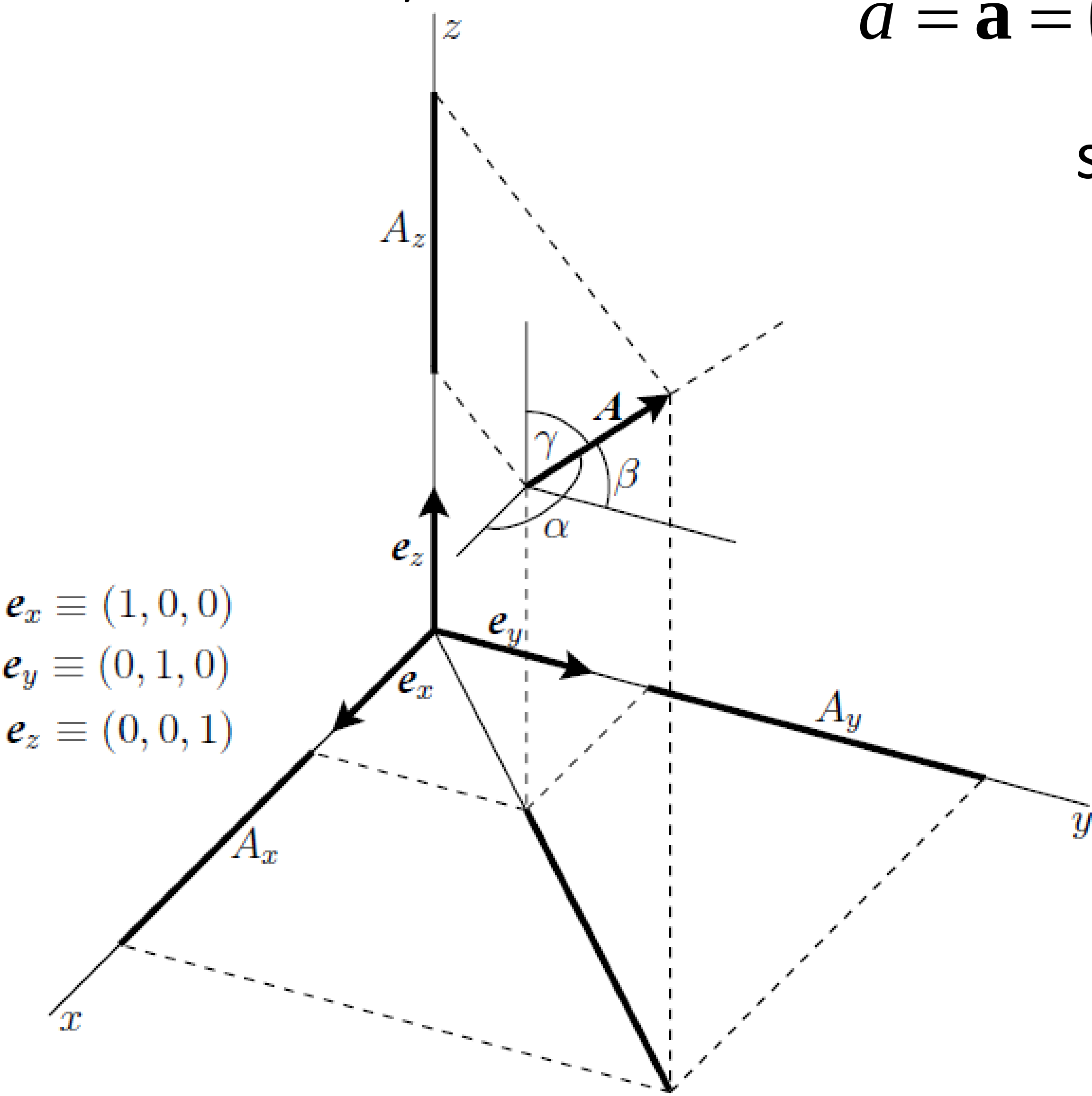
tensors (generalisation of a vector – quantity has several dimensions) $\overline{\mathbf{T}}$

(tensor of tension,...)

Vectors – basic properties and operations:

Vector is given by its components (in Cartesian coordinate system a_x, a_y, a_z):

$$\vec{a} = \mathbf{a} = \left(a_x, a_y, a_z\right)$$



size of the vector:

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

directional angles of the vector: α, β, γ

$$a_x = |\mathbf{a}| \cos\alpha$$

$$a_y = |\mathbf{a}| \cos\beta$$

$$a_z = |\mathbf{a}| \cos\gamma$$

Vectors – basic properties and operations:

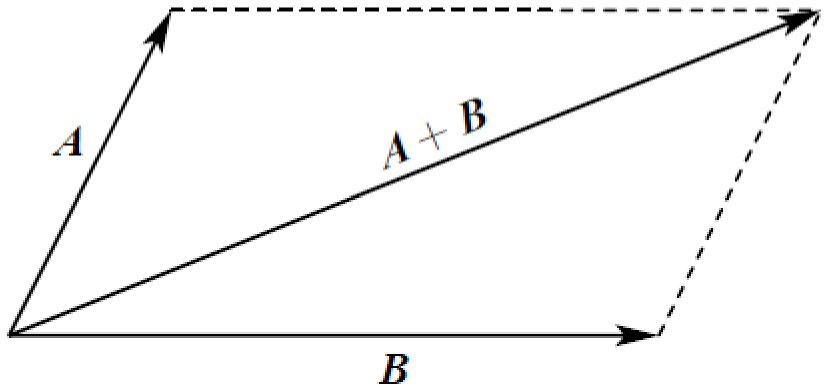
Multiplication of a vector **A** by a scalar f:

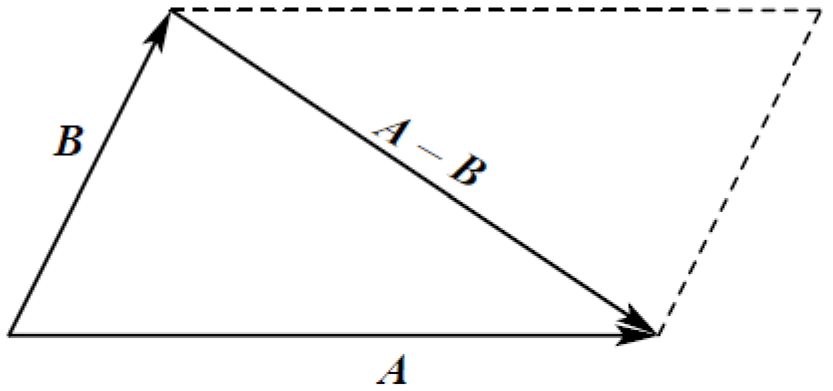
$$\boldsymbol{A} \cdot f = f \cdot \boldsymbol{A} = (f\, A_x \boldsymbol{i} + f\, A_y \boldsymbol{j} + f\, A_z \boldsymbol{k})$$

Addition and subtraction of two vectors (**A** and **B**):

$$\boldsymbol{A} \pm \boldsymbol{B} = (A_x \pm B_x)\,\boldsymbol{i} + (A_y \pm B_y)\,\boldsymbol{j} + (A_z \pm B_z)\,\boldsymbol{k}$$

graphical way...





Vectors – basic properties and operations:

Scalar multiplication of a vector **A** and **B**:

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$$

or (dot product)

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos \vartheta$$

where ϑ is the angle between these two vectors (**A** and **B**).

Result of this operation is a scalar (number).

When these vectors are orthogonal (angle between them is 90°), then scalar multiplication is equal to zero.

Comment: Scalar multiplication is commutative operation.

Vectors – basic properties and operations:

Vector multiplication of a vector **A** and **B**:

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{e}_x + (A_z B_x - A_x B_z)\boldsymbol{e}_y + (A_x B_y - A_y B_x)\boldsymbol{e}_z$$

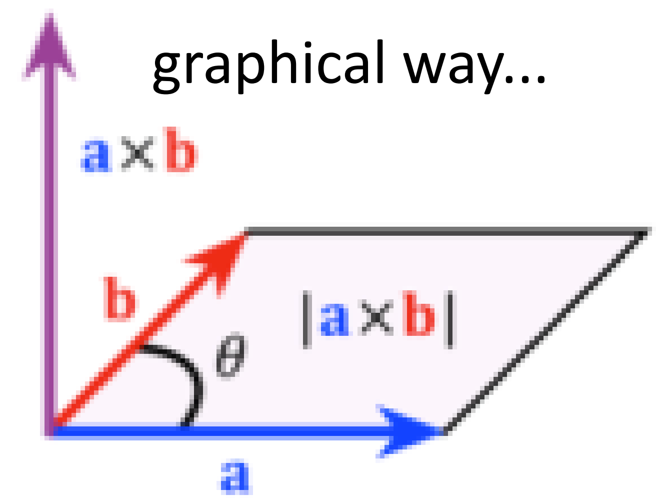
and (cross product)

$$|\boldsymbol{C}| = |\boldsymbol{A}|\,|\boldsymbol{B}|\sin\vartheta$$

where ϑ is the angle between these two vectors (**A** and **B**).

Result of this operation is a vector.

graphical way...



Comment: Vector multiplication is anti-commutative operation.

$$\boldsymbol{B} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{B}$$

Vectors – basic properties and operations:

Vector multiplication of a vector **A** and **B**:

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{e}_x + (A_z B_x - A_x B_z)\boldsymbol{e}_y + (A_x B_y - A_y B_x)\boldsymbol{e}_z$$

(cross product)

We can express this formula in a very compact form – as a determinant with 3 × 3 elements:

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vectors – basic properties and operations:

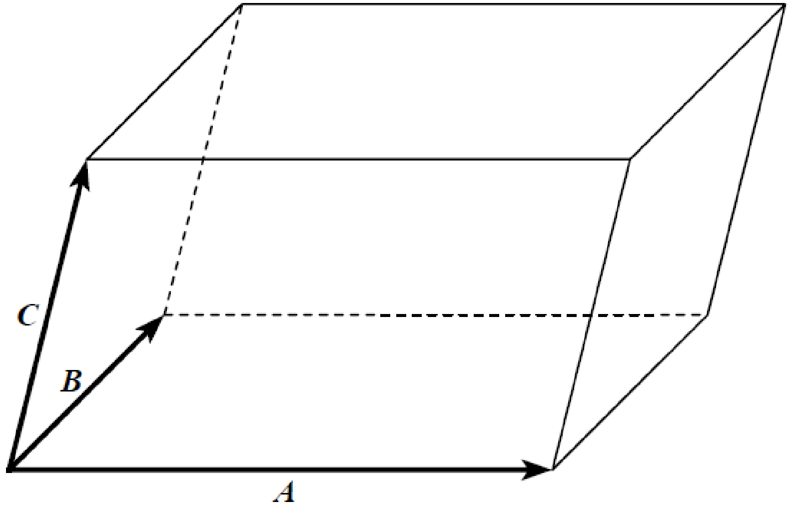
Mixed multiplication (triple product) of a vector **A** , **B** and **C**:

$$\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}.$$

$$\begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$

Result of this operation is a scalar (number).

It is the volume of a paralelipiped with a base given by **A** and **B** vectors and **C** is connected with its height.



Vectors – basic properties and operations:

Double vector multiplication of vectors **A** , **B** and **C**:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

Result of this operation is a vector.

Vectors – basic properties and operations

Linear combination of vectors

$$\mathbf{v} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 + \cdots \alpha_n \mathbf{a}_n$$

$$\boldsymbol{e}_x \equiv (1, 0, 0)$$
$$\boldsymbol{e}_y \equiv (0, 1, 0)$$
$$\boldsymbol{e}_z \equiv (0, 0, 1)$$

$$\mathbf{v} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

Linearly dependent vectors

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 + \cdots \alpha_n \mathbf{a}_n = \mathbf{0}$$

where not all α_i are zero