Mathematics for Biochemistry

LECTURE 17

Matrices 1

Content:

- Definition
- Operations
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Matrix:

Rectangular array (table) of numbers, symbols or expressions.

Given by number of **rows (m)** and **columns (n)** → $\mathbf{A}_{mn}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 6 \end{pmatrix}$$

A is 2×3 matrix

Row vector is 1×n matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Column vector is m×1 matrix

$$\mathbf{C} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

Entries of Matrix: a_{ij} or $a_{i,j}$

$$\mathbf{A} = \left(a_{ij} \right) \in \mathbb{R}^{m \times n}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & \boxed{6} \end{pmatrix}$$

$$a_{23} = \boxed{6}$$

Main diagonal – list of all entries $\mathbf{a_{ij}}$ where **i = j**

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & -5 \\ 2 & 3 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 8 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 5 & -6 \\ 4 & 2 & 5 \\ 1 & 3 & 8 \end{pmatrix}$$

Trace – sum of all entries $\mathbf{a_{ij}}$ where **i = j**

$$Tr(\mathbf{C}) = 1 + 2 + 8 = 11$$

Square matrix is n×n matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 5 & -6 \\ 4 & 2 & 5 \\ 1 & 3 & 8 \end{pmatrix}$$

Identity matrix is n×n matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank – maximal number of linearly independent rows or columns

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank (I) = 3

$$\mathbf{F} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Rank (F) = 1

Submatrix – obtained by deleting collection of rows and/or columns

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 8 & 5 \\ 4 & 9 & 8 & 6 & 2 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 6 & 8 & 5 \\ 4 & 8 & 6 & 2 \end{pmatrix}$$

Operations with matrices:

Addition

$$\mathbf{A} \pm \mathbf{B} = a_{ij} \pm b_{ij} = (a \pm b)_{ij} \; ; \qquad 1 \le i \le m \quad 1 \le j \le n$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 4 & -6 & 9 \end{pmatrix} + \begin{pmatrix} 5 & -4 & 6 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -2 & 11 \\ 5 & -4 & 12 \end{pmatrix}$$

Scalar multiplication

$$c\mathbf{A} = c \cdot a_{ij} = (ca)_{ij} \; ; \qquad 1 \le i \le m \quad 1 \le j \le n$$

$$-2 \cdot \begin{pmatrix} 1 & 2 & 5 \\ 4 & -6 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -10 \\ -8 & 12 & -18 \end{pmatrix}$$

Operations with matrices:

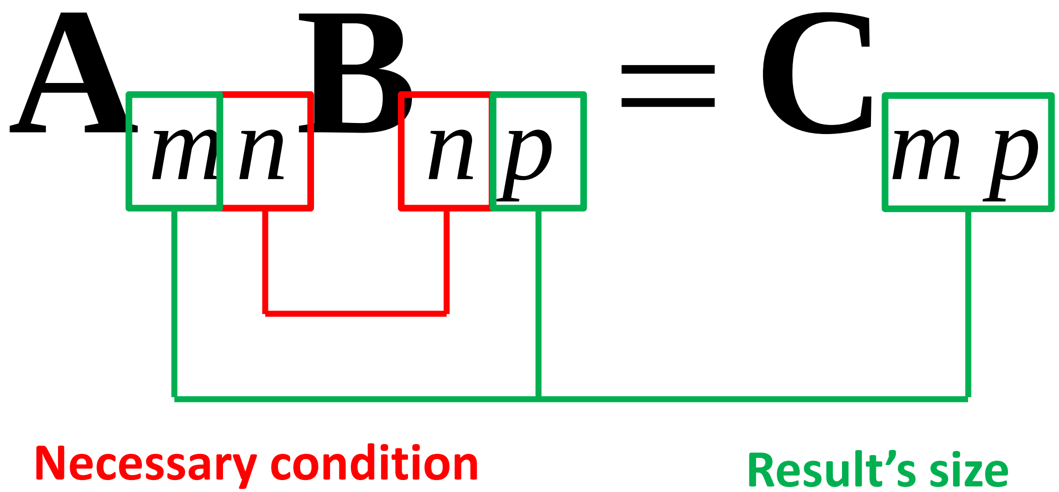
Transposition – transpose of **A** is matrix $\mathbf{A}^{\mathrm{T}}$

$$(a)_{ij}^{T} = (a)_{ji}; \qquad 1 \le i \le m \quad 1 \le j \le n$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ 4 & -6 & 9 \end{pmatrix} \rightarrow \mathbf{A}^{T} = \begin{pmatrix} 1 & 4 \\ 2 & -6 \\ 5 & 9 \end{pmatrix}$$

Rows became columns and vice versa

Matrix multiplication



$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for i = 1,…,m and j = 1,…,p

In words: the resultant entry c_{ij} is obtained by **scalar multiplication** of **ith row** with **jth column**

Matrix multiplication is not (in general) commutative operation: AB ≠ BA

Example

$$\begin{pmatrix} 1 & 2 \\ -4 & 5 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix} =$$

3x2 2x4

$$= \begin{pmatrix} 1\cdot 2+2\cdot 0 & 1\cdot(-2)+2\cdot 1 & 1\cdot 1+2\cdot 2 & 1\cdot 3+2\cdot 4 \\ -4\cdot 2+5\cdot 0 & -4\cdot(-2)+5\cdot 1 & -4\cdot 1+5\cdot 2 & -4\cdot 3+5\cdot 4 \\ 1\cdot 2+(-3)\cdot 0 & 1\cdot(-2)+(-3)\cdot 1 & 1\cdot 1+(-3)\cdot 2 & 1\cdot 3+(-3)\cdot 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 0 & 5 & 11 \\ -8 & 13 & 6 & 8 \\ 2 & -5 & -5 & -9 \end{pmatrix}$$

Inverse matrix is n×n **matrix** such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

If **B** is inverse to **A**, it usually denoted as **A**$^{-1}$ (**B** = **A**$^{-1}$)

Some properties

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$\left(k\mathbf{A}\right)^{-1} = k^{-1}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T}$$

$$\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$det\left(\mathbf{A}^{-1}\right) = \left(det\,\mathbf{A}\right)^{-1}$$

Determinant of **n×n matrix** is a scalar value characterizing some properties of matrix

2×2

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow det(\mathbf{M}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3×3

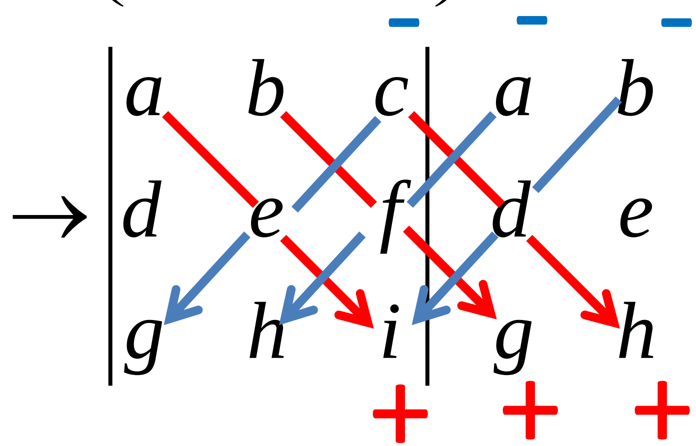
$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow det(\mathbf{M}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$$

$$= aei - afh + bfg - bdi + cdh - ceg$$

Rule of Sarrus

3×3

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow det(\mathbf{M}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow$$



$$= aei - afh + bfg - bdi + cdh - ceg$$

Expansion with respect to the given row

4×4

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \to det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$=(-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} +$$

$$+(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + (-1)^{1+4} a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Expansion with respect to the given row

Example

4×4

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 0 & 2 & 0 & 1 \\ 2 & 4 & -2 & 1 \\ 1 & 0 & -5 & 2 \end{pmatrix} \to det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & -4 & 5 \\ 0 & 2 & 0 & 1 \\ 2 & 4 & -2 & 1 \\ 1 & 0 & -5 & 2 \end{vmatrix} =$$

$$= (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 2 & -4 & 5 \\ 4 & -2 & 1 \\ 0 & -5 & 2 \end{vmatrix} + (-1)^{2+2} \cdot 2 \cdot \begin{vmatrix} 1 & -4 & 5 \\ 2 & -2 & 1 \\ 1 & -5 & 2 \end{vmatrix} + (-1)^{2+3} \cdot 0 \cdot \begin{vmatrix} 1 & 2 & 5 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \end{vmatrix} + (-1)^{2+4} \cdot 1 \cdot \begin{vmatrix} 1 & 2 & -4 \\ 2 & 4 & -2 \\ 1 & 0 & -5 \end{vmatrix} =$$

$$= 2 \cdot \begin{vmatrix} 1 & -4 & 5 \\ 2 & -2 & 1 \\ 1 & -5 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & -4 \\ 2 & 4 & -2 \\ 1 & 0 & -5 \end{vmatrix} = -54 + 12 = -42$$