Mathematics for Biochemistry

LECTURE 18

Matrices 2

Content:

- Inverse matrix
- Elimination method
- SLAE

Inverse matrix is n×n matrix such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

If **B** is inverse to **A**, it usually denoted as $\mathbf{A}^{-1}$ **(B = $\mathbf{A}^{-1}$)**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightarrow \mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\mathbf{AA}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Lower triangular matrix is n×n matrix such that:

$$\mathbf{L} = \begin{pmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & \cdots & \cdots & 0 \\ \vdots & l_{32} & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & l_{nn} \end{pmatrix}$$

Upper triangular matrix is n×n matrix such that:

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1,n} \\ 0 & u_{22} & \cdots & \cdots & u_{2,n} \\ \vdots & 0 & \ddots & \cdots & u_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & u_{nn} \end{pmatrix}$$

Gauss – Jordan elimination method

- Swap position of two rows
- Multiply a row with non-zero number
- Add one row to (number multiple) another

$$\begin{pmatrix} 1 & 5 & -6 \\ 4 & 2 & 5 \\ 1 & 3 & 8 \end{pmatrix} \cdot(-1)\,/\ add\ to\ 3^{rd}\ \ row \Big|\cdot(-4)\,/\ add\ to\ 2^{nd}\ \ row \quad \sim$$

$$\begin{pmatrix} 1 & 5 & -6 \\ 0 & -18 & 29 \\ 0 & -2 & 14 \end{pmatrix} \cdot(-9)\,/\ add\ to\ 2^{nd}\ \ row \sim \begin{pmatrix} 1 & 5 & -6 \\ 0 & 0 & -97 \\ 0 & -2 & 14 \end{pmatrix} swap\ second\ and\ third\ row$$

$$\begin{pmatrix} 1 & 5 & -6 \\ 0 & -2 & 14 \\ 0 & 0 & -97 \end{pmatrix} \begin{matrix} \\ /(-2) \\ /(-97) \end{matrix} \sim \begin{pmatrix} 1 & 5 & -6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{pmatrix}$$

This procedure can continue until the unit matrix is obtained

Gauss – Jordan elimination method

$$\begin{pmatrix} 1 & 5 & -6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{pmatrix} \cdot 7 \,/\, \textit{add to the } 2^{nd}\textit{ row} \Big| \cdot (6) \,/\, \textit{add to the } 1^{st}\textit{ row} \quad \sim$$

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot (-5) \,/\, \textit{add to the } 1^{st}\textit{ row} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The result of GJEM is the unit matrix → the **rank** of given matrix (the number of linearly independent rows) is **3**.

Gauss – Jordan elimination method in inverse matrix calculation

$$\left(\underbrace{\begin{matrix}1 & 5 & -6\\ 4 & 2 & 5\\ 1 & 3 & 8\end{matrix}}_{\text{original matrix}}\left|\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{\text{unit matrix}}\right.\right)\cdot\left(-4\right)/\ to\ 2^{nd}\ \Big|\cdot\left(-1\right)/\ to\ 3^{rd}\qquad\sim$$

$$\left(\begin{array}{ccc|ccc}1 & 5 & -6 & 1 & 0 & 0\\ 0 & -18 & 29 & -4 & 1 & 0\\ 0 & -2 & 14 & -1 & 0 & 1\end{array}\right)\cdot\left(-9\right)/\ to\ 2^{nd}\ \Big|\cdot\left(2\right)/\ to\ 1^{st}\ \Big|/\left(-2\right)\qquad\sim$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 22 & -1 & 0 & 2\\ 0 & 0 & -97 & 5 & 1 & -9\\ 0 & 1 & -7 & \frac{1}{2} & 0 & -\frac{1}{2}\end{array}\right)swap\ with\ 2^{nd}\qquad\sim$$

Gauss – Jordan elimination method in inverse matrix calculation

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 22 & -1 & 0 & 2 \\ 0 & 1 & -7 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & -97 & 5 & 1 & -9 \end{array}\right) \begin{array}{l} \cdot(-1)/\ \text{add to } 1^{st} \\ /(-97) \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 29 & -\frac{3}{2} & 0 & \frac{5}{2} \\ 0 & 1 & -7 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{97} & -\frac{1}{97} & \frac{9}{97} \end{array}\right) \cdot(-29)/\ \text{add to } 1^{st} \,\Big|\, \cdot(7)/\ \text{add to } 2^{nd} \sim$$

$$\left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{unit matrix}} & \underbrace{\begin{matrix} -\frac{1}{194} & \frac{29}{97} & -\frac{37}{194} \\ \frac{27}{194} & -\frac{7}{97} & \frac{29}{194} \\ -\frac{5}{97} & -\frac{1}{97} & \frac{9}{97} \end{matrix}}_{\text{inverse matrix}} \end{array}\right)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{194} & \frac{29}{97} & -\frac{37}{194} \\ \frac{27}{194} & -\frac{7}{97} & \frac{29}{194} \\ -\frac{5}{97} & -\frac{1}{97} & \frac{9}{97} \end{pmatrix} = \frac{1}{194} \begin{pmatrix} -1 & 58 & -37 \\ 27 & -14 & 29 \\ -10 & -2 & 18 \end{pmatrix}$$

SLAE – System of Linear Algebraic Equations

$$\begin{aligned} x + 2y &= 4 \\ x - y &= 5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= -2 \\ -3x_1 - x_2 - 2x_3 &= -4 \end{aligned}$$

SLAE – Matrix Form

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -3 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$$

The solution is **unique** if and only if the **rank** of the coefficient matrix is the **same** as number of **rows.**

SLAE – GJEM

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & -2 \\ \underbrace{-3 \quad -1 \quad -2}_{coefficient\ matrix} & & & \underset{right\ hand\ side}{-4} \end{array}\right) \cdot(-2)\ to\ 2^{nd}\ /\cdot(3)\ to\ 3^{rd} =$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & -4 \\ 0 & 5 & 7 & -1 \end{array}\right) \cdot(-5)\ to\ 3^{rd} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 22 & 19 \end{array}\right)$$

$$22x_3 = 19 \Rightarrow x_3 = \frac{19}{22}; \quad x_2 - 3\left(\underset{x_3}{\frac{19}{22}}\right) = -4 \Rightarrow x_2 = -\frac{31}{22}; \quad x_1 + 2\left(\underset{x_2}{-\frac{31}{22}}\right) + 3\underset{x_3}{\frac{19}{22}} = 1 \Rightarrow x_1 = \frac{27}{22}$$

$$\mathbf{x}^T = \frac{1}{22}(27, -31, 19)$$

SLAE – Inverse matrix

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -3 & -1 & -2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}}_{\mathbf{B}}$$

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{B} \, / \, \mathbf{A}^{-1}$$

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{I}\cdot\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

$$\underline{\underline{\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}}}$$

$$\underbrace{\begin{pmatrix} -\frac{7}{22} & \frac{1}{22} & -\frac{9}{22} \\ -\frac{5}{22} & \frac{7}{22} & \frac{3}{22} \\ \frac{13}{22} & -\frac{5}{22} & \frac{1}{22} \end{pmatrix}}_{\mathbf{A}^{-1}} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -3 & -1 & -2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{X}} = \underbrace{\begin{pmatrix} -\frac{7}{22} & \frac{1}{22} & -\frac{9}{22} \\ -\frac{5}{22} & \frac{7}{22} & \frac{3}{22} \\ \frac{13}{22} & -\frac{5}{22} & \frac{1}{22} \end{pmatrix}}_{\mathbf{A}^{-1}} \underbrace{\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}}_{\mathbf{B}}$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{X}} = \frac{1}{22} \underbrace{\begin{pmatrix} 27 \\ -31 \\ 19 \end{pmatrix}}_{\mathbf{A}^{-1}\mathbf{B}}$$

SLAE – Cramer's rule

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -3 & -1 & -2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}}_{\mathbf{B}} \rightarrow x_i = \frac{det(\mathbf{A}_i)}{det(\mathbf{A})}$$

$$det(\mathbf{A}_1) = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 5 & 3 \\ -4 & -1 & -2 \end{vmatrix} = 27$$

$$det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -3 & -1 & -2 \end{vmatrix} = 22$$

$$det(\mathbf{A}_2) = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 3 \\ -3 & -4 & -2 \end{vmatrix} = -31$$

$$det(\mathbf{A}_3) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -2 \\ -3 & -1 & -4 \end{vmatrix} = 19$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 27 \\ -31 \\ 19 \end{pmatrix}$$

Important problems

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$$\begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0.01 \\ 0.010001 \end{pmatrix}$$

$$\begin{pmatrix} 1.000001 & 1 \\ 2 & 2.000001 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2.000001 \\ 4.000001 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1.000001 & 1 \\ 2 & 2.000001 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2.001001 \\ 4.000001 \end{pmatrix}$$