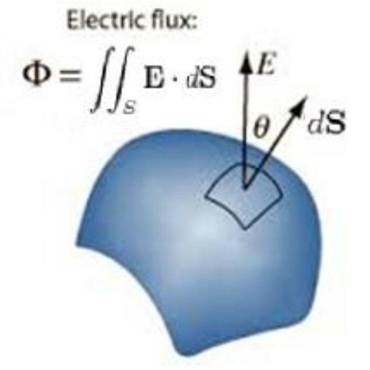
Topic 9: electromagnetism – Maxwell‘s equations

Content:

- summary of known EM-laws and math. formalism
- Maxwell‘s equations in integral and differential form
- EM waves, Poynting vector

- **summary of known EM-laws and math. formalism**

Lecture Nr. 5 (electricity, WT): slide nr. 23:



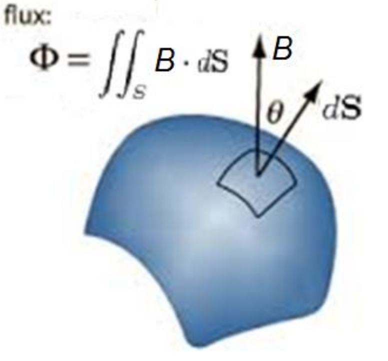
1.

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

Gauss's law for electric field

(is non-zero due to the monopolar character of electric field)

Lecture Nr. 6 (magnetism): slide nr. 32:



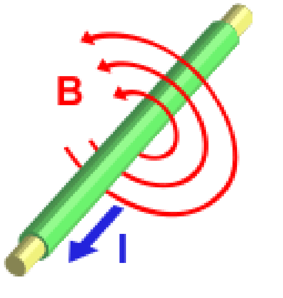
2.

$$\Phi_B = \oiint_S \vec{B} \cdot d\vec{S} = 0$$

Gauss's law for magnetic field

(flux is zero due to the dipole character of magnetic field)

Lecture Nr. 8 (electro-magnetism): slide nr. 10:



3.

$$\oint_{l(S)} \vec{B} \cdot d\vec{l} = \mu_0 I$$

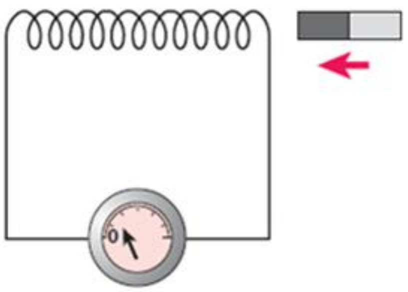
Ampere's law

(integration of magnetic induction along a closed circle – so called circulation)

- **summary of known EM-laws and math. formalism**

Lecture Nr. 8 (electro-magnetism): slide nr. 12:

4. $$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\iint_S \vec{B}\cdot d\vec{S}\right]$$



Faraday‘s law of induction

(due to the dipole character of magnetic field)

(here ε is not electric permitivity, but electromotive force is in [V]).

Electromotive force is the voltage developed by any source of electrical energy.

It can be also evaluated by means of the circulation integral for the electric field:

$$\varepsilon = \int_A^B \vec{E}\cdot d\vec{l}$$

so, we can write for the Farraday’s law of induction:

$$\oint_{l(S)} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$

Maxwell’s equations – integral form

These 4 equations together with Lorentz force law (lecture Nr. 8, slide nr. 6) form the foundation of classical electrodynamics, classical optics, and electric circuits.

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Electric force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{Magnetic force}}$$

Lorentz force law

The 4 basic equations are in general called as Maxwell’s equations.

They are named after the physicist and mathematician **James Clerk Maxwell**, who published an early form of those equations between 1861 and 1862.

James Clerk Maxwell (1831–1879)

With the publication of A Dynamical Theory of the Electromagnetic Field in 1865, Maxwell demonstrated that electric and magnetic fields travel through space as waves moving at the speed of light.

Maxwell's equations for electromagnetism have been called the "second great unification“ in physics (the first one was from Isaac Newton).

Maxwell’s equations – integral form

Formulation of Maxwell‘s equations (ME) is connected with the development of physics in the end of 19th century, when the internal structure of matter was not well known (idea about the existence of positive and negative charges in their structure was accepted), so they are based mostly on the description of macroscopic phenomena.

From the point of view of mathematical formalism, we divide ME in their integral and differential form. In the actual stage of this lecture, we have started with the integral form.

Maxwell's equations – integral form

$$\oiint_{S(V)} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho dV$$ Gauss law

$$\oiint_{S(V)} \vec{B} \cdot d\vec{S} = 0$$ Gauss law for magnetism

$$\oint_{l(S)} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S}$$ Maxwell-Faraday equation

$$\oint_{l(S)} \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{J} \cdot d\vec{S} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{S}$$ Maxwell-Ampere equation

where: *V* is volume, enclosed by surface *S(V)*, *t* is time,

$l(S)$ is curve enclosing a surface *S*,

E is electrical field intensity vector, **B** is magnetic induction vector,

ρ is volume density of electrical charge,

J is density of electrical current (vector quantity),

ε_0 is electrical permitivity of vacuum, μ_0 is magnetic permeability of vacuum

Maxwell's equations – integral form

The 4 basic equations are in general called as Maxwell's equations.

Name	Integral equations	Meaning
Gauss's law	$\oiint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$	The electric field leaving a volume is proportional to the charge inside.
Gauss's law for magnetism	$\oiint_S \mathbf{B}\cdot d\mathbf{S} = 0$	There are no magnetic monopoles; the total magnetic flux piercing a closed surface is zero.
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_\ell \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S}$	The voltage accumulated around a closed circuit is proportional to the time rate of change of the magnetic flux it encloses.
Ampère's circuital law (with Maxwell's addition)	$\oint_\ell \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \iint_S \mathbf{J}\cdot d\mathbf{S} + \mu_0\varepsilon_0 \frac{d}{dt}\iint_S \mathbf{E}\cdot d\mathbf{S}$	Electric currents and changes in electric fields are proportional to the magnetic field circulating about the area they pierce.

Instead of current *I* and charge *Q*, there are used integrals of current density (***J***) and charge density (*ρ*):

$$I = \iint_S \vec{J}\cdot d\vec{S} \qquad\qquad Q = \iiint_V \rho dv$$

In the last equation (Amper's law) there is an additional term (added by Maxwell):

$$\mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0\varepsilon_0 \frac{d}{dt}\iint_S \vec{E}\cdot d\vec{S}$$

called therefore also as Maxwell-Ampere equation

Maxwell's equations in vacuum (free space)

Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial \Sigma} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_{\Sigma} \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Sigma} \mathbf{E} \cdot \mathrm{d}\mathbf{S}$

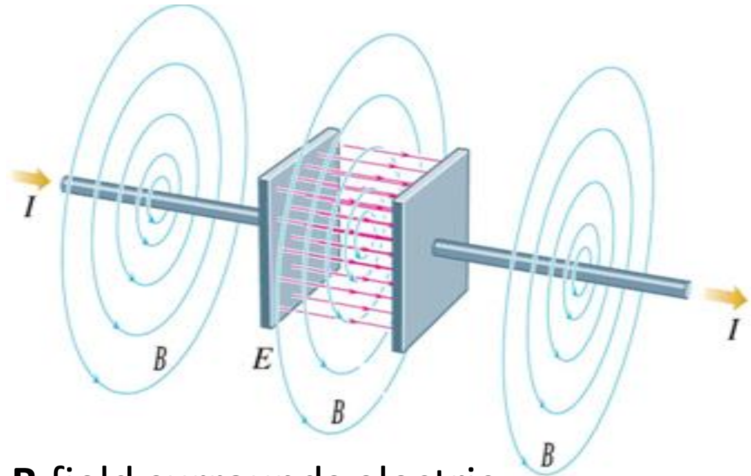
In vacuum there are no free charges and no current ($\rho = 0$, $\mathbf{J} = 0$).

$$\oiint_{S(V)} \vec{E} \cdot d\vec{S} = 0$$

$$\oiint_{S(V)} \vec{B} \cdot d\vec{S} = 0$$

$$\oint_{l(S)} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S} \vec{B} \cdot d\vec{S}$$

$$\oint_{l(S)} \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \iint_{S} \vec{E} \cdot d\vec{S}$$



B field surrounds electric field **E** (capacitor in the central part of the figure), although there is no "current" flowing here

Here the role of the added term by Maxwell is clearly visible (without it, it would not be possible the explain the situation of **E** field acting in vacuum). The term $\varepsilon_0\, d\Phi_E/dt$ is sometimes called as displacement current.

Videos with experiments: https://www.youtube.com/watch?v=EqufEpWaKXw

Excellent lecture from Walter Lewin: https://www.youtube.com/watch?v=8ZYFYUFRblM

Maxwell’s equations – integral form

Oliver Heaviside
1850 - 1925

We have this common form thanks to the work of Oliver Heaviside - he was able to rewrite Maxwell's original 20 equations into a mathematically equivalent 4 equation form.

Maybe we should call them as Maxwell-Heaviside’s equations (?).

Repetition: first two equations describe how the fields vary in space due to sources if any; electric fields emanating from electric charges in Gauss's law, and magnetic fields as closed field line in Gauss's law for magnetism.
The other two describe how the fields "circulate" around their respective sources; the magnetic field "circulates" around electric currents and time varying electric fields in Ampere's law with Maxwell's addition, while the electric field "circulates" around time varying magnetic fields in Faraday's law.

Maxwell's equations – differential form

Name	Integral equations	Differential equations
Gauss's law	$\oiint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\oiint_S \mathbf{B}\cdot d\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_\ell \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_\ell \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0\iint_S \mathbf{J}\cdot d\mathbf{S} + \mu_0\varepsilon_0\frac{d}{dt}\iint_S \mathbf{E}\cdot d\mathbf{S}$	$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$

To transfer from the integral form to that – differential one is not easy, we have to know special properties of *div* and *rot* differential operators and so called Stokes theorem – we will not perform it in detail here...

There are several good web-sites and also videos about this, e.g.:
https://www.wikihow.com/Convert-Maxwell%27s-Equations-into-Differential-Form

Maxwell's equations – differential form

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ Gauss law

$$\nabla \cdot \vec{B} = 0$$ Gauss law for magnetism

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Maxwell-Faraday equation

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$ Maxwell-Ampere equation

where: **E** is electrical field intensity vector, **B** is magnetic induction vector,
ρ is volume density of electrical charge,
J is density of electrical current (vector quantity),
ε_0 is electrical permitivity of vacuum, μ_0 is magnetic permeability of vacuum

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In some textbooks instead of **E** field the electric induction field **D** is used (**D** = ε_0 **E**). In such a case, we can write the Gauss law:

$$\nabla \cdot \vec{D} = \rho$$

Maxwell's equations – differential form (vacuum, empty space)

$$\nabla \cdot \vec{E} = 0$$

Gauss law

$$\nabla \cdot \vec{B} = 0$$

Gauss law for magnetism

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell-Faraday equation

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell-Ampere equation

where: **E** is electrical field intensity vector, **B** is magnetic induction vector,
ρ is volume density of electrical charge,
J is density of electrical current (vector quantity),
ε_0 is electrical permitivity of vacuum, μ_0 is magnetic permeability of vacuum

Maxwell’s equations

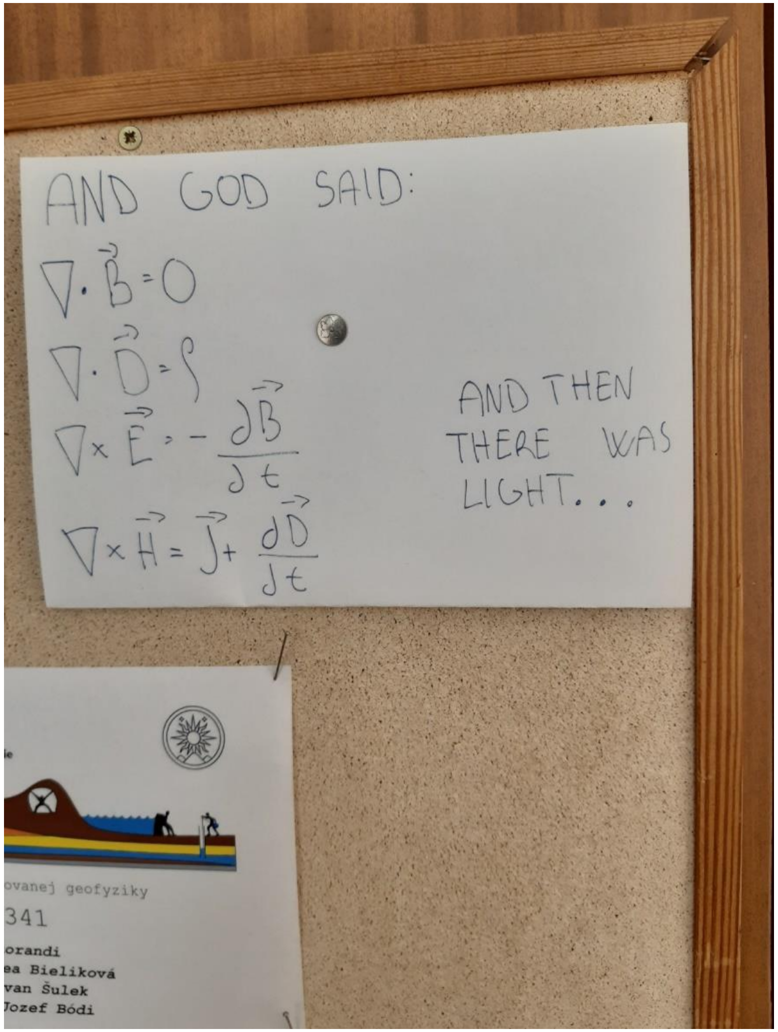
... these are so popular ;-)

were plotted on a wall
close to our faculty...

... and on the ground,
In the front of technical university

Maxwell's equations

... these are so popular ;-)



... even in our hallway

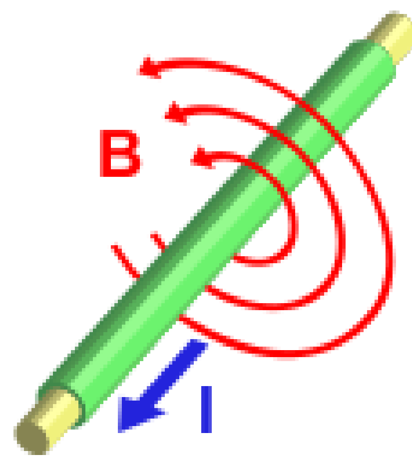
...you can purchase them as T-shirts...

Maxwell's equations

One very important contribution from Maxwell – symmetry (between electric and magnetic fields):

- a time varying magnetic field produces an electric field,
- a time varying electric field produces a magnetic field.

This symmetry is fully valid thanks to the additional term from Maxwell to the Ampere's equation.



EM waves

Electromagnetic waves (EM radiation) are synchronized oscillations of electric and magnetic fields that propagate at the speed of light through a vacuum.

Faraday's law: **dB/dt → electric field**

Maxwell's modification of Ampere's law

dE/dt → magnetic field

$$\oint_{l(S)} \vec{E} \cdot d\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{B} \cdot d\vec{S} \qquad \oint_{l(S)} \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{E} \cdot d\vec{S}$$

These two equations can be solved simultaneously.

The result is:

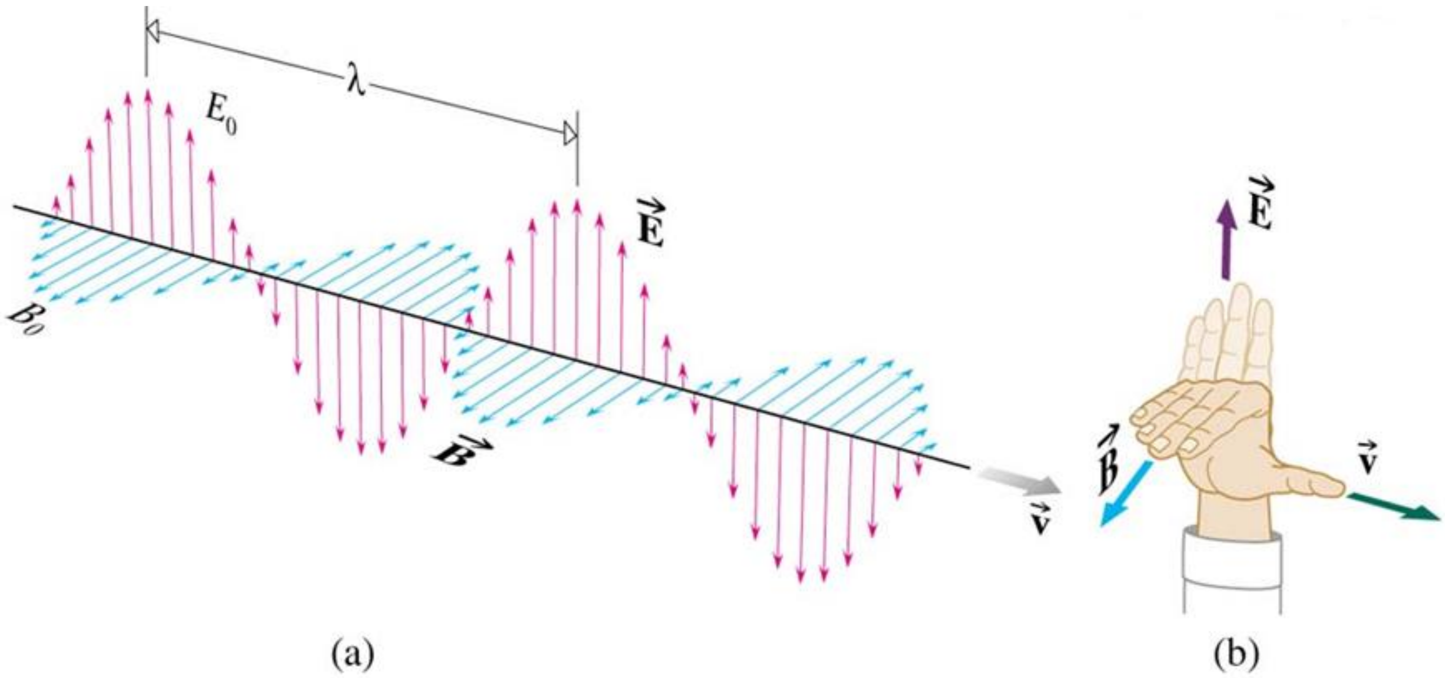
$$\mathbf{E}(x, t) = E_P \sin(kx - \omega t)\ \hat{\mathbf{j}}$$

$$\mathbf{B}(x, t) = B_P \sin(kx - \omega t)\ \hat{\mathbf{k}}$$

EM waves

One important consequence of the EM symmetry – origin of EM waves: changing electric and magnetic fields create a wave:

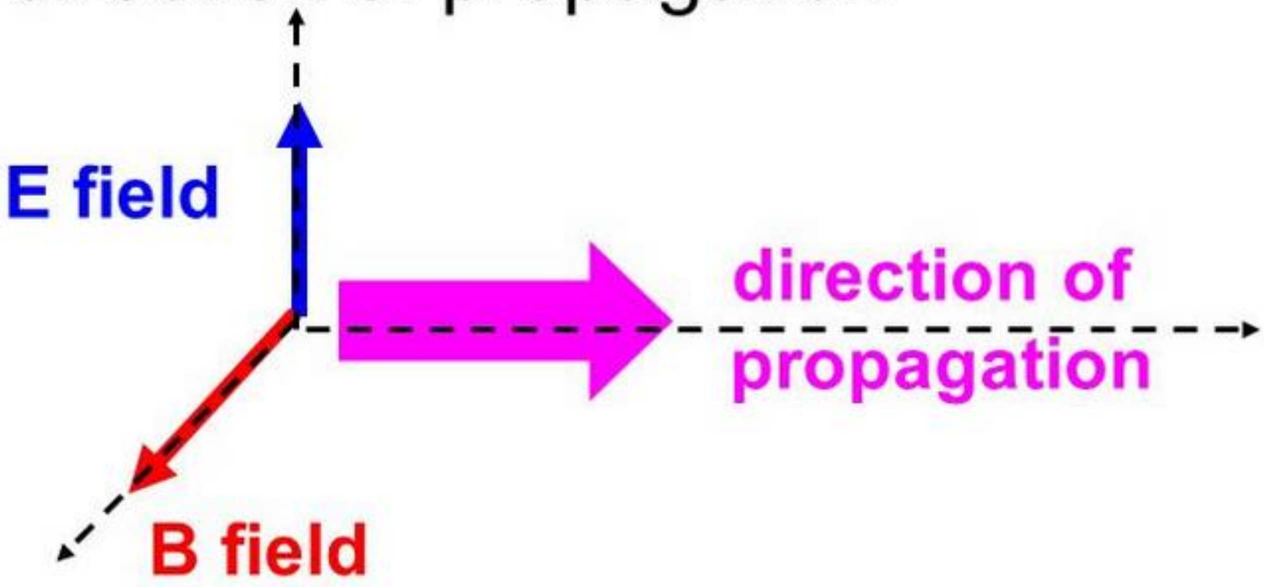
- electric field creates a magnetic field
- magnetic field creates an electric field



Electromagnetic waves are produced whenever charged particles are accelerated, and these waves can subsequently interact with any charged particles. EM waves carry energy.

EM waves

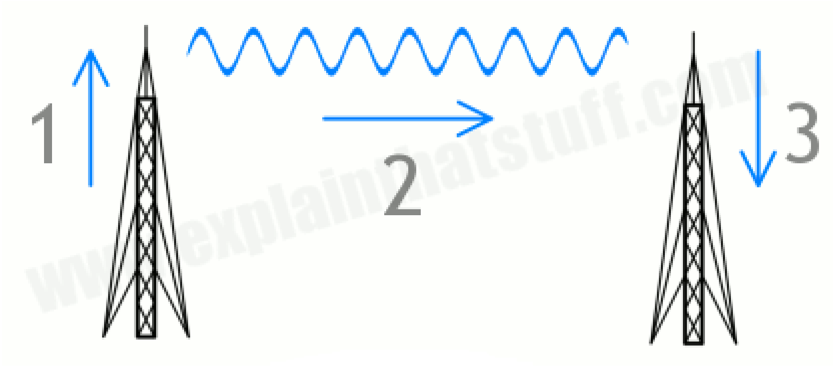
- the electromagnetic wave is a *transverse wave*, the electric and magnetic fields oscillate in the direction perpendicular to the direction of propagation



good visualisation:
https://en.wikipedia.org/wiki/Electromagnetic_radiation#/media/File:Electromagneticwave3D.gif

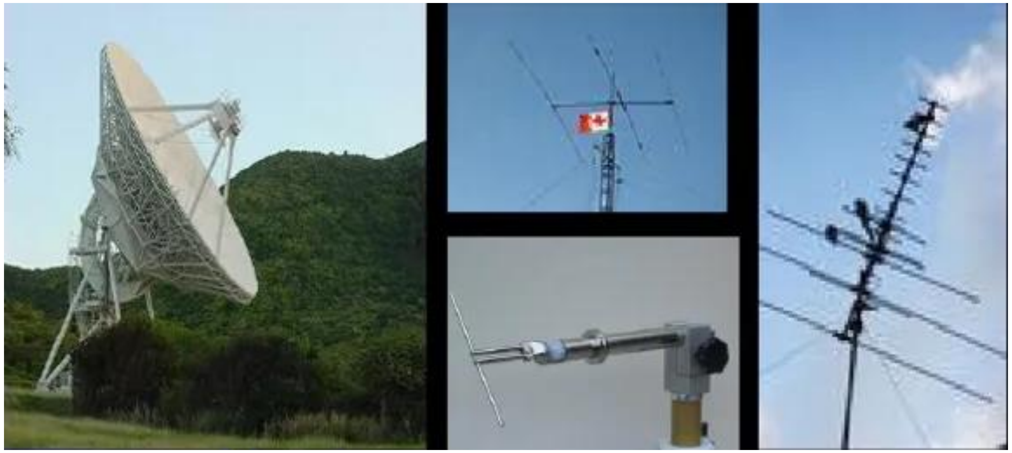
EM waves

in practical life – performed by means of antennas



transmitter antenna

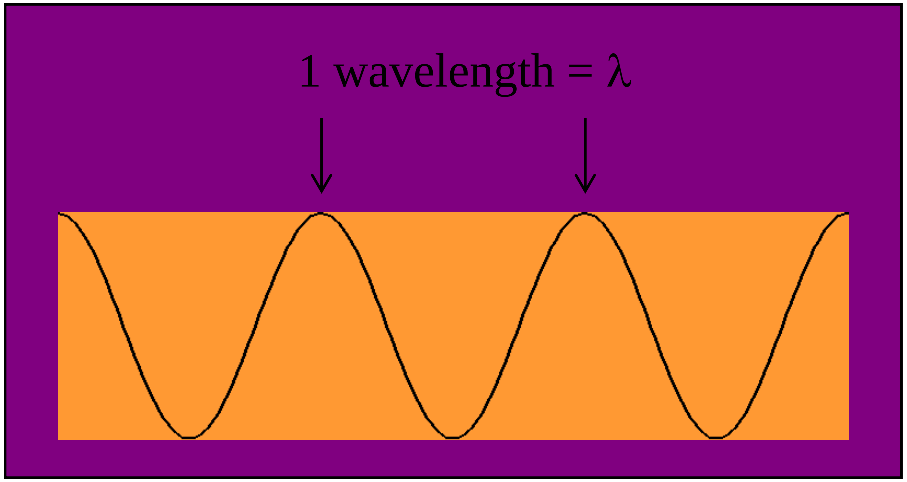
receiver antenna



EM waves

- for a continuous wave the **speed** *v* is the **wavelength** compared to the **period** (reciprocal frequency):

$$v = \lambda / T = \lambda f$$



- for an electromagnetic wave the speed is based on the permittivity and permeability,

$$v = \sqrt{1/\mu\varepsilon}$$

$$c = \sqrt{1/\mu_0\varepsilon_0}$$

- in the vacuum this is the speed of light c = 2.99792 $\cdot 10^8$ m/s.

Task: check the value of c by entering values for electric permitivity of vacuum ε_0 and magnetic permeability of vacuum μ_0.

EM waves – derivation of velocity

Faraday law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Ampére law

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

Now if you look carefully, you'll see that one term in each equation equals zero and the other can be replaced with a time derivative.

$$0 - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \qquad 0 - \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

Let's clean it up a bit and see what we get.

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} \qquad \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{B}$$

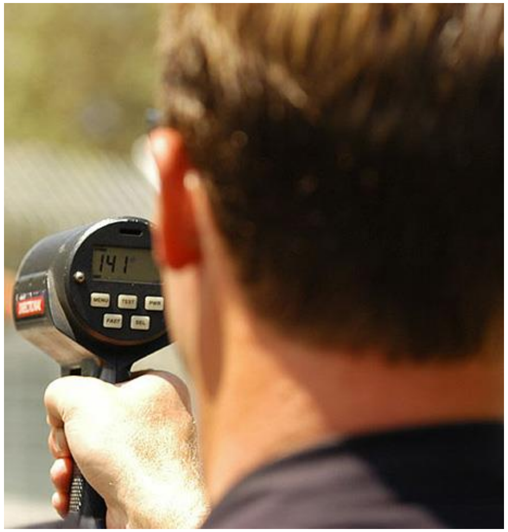
If you compare this equation to the mechanical wave equation: $\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 D}{\partial t^2}$

Then it would be logical to define the speed of an electromagnetic wave to be

$v_{EM-wave} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 300000 km/s = c$.

EM waves

- All EM waves travel 300,000 km/sec in vacuum (speed of light-nature's limit!).
- EM waves usually travel slowest in solids and fastest in gases.



Material	Speed (km/s)
Vacuum	300,000
Air	<300,000
Water	226,000
Soil	100,000
Ice	150,000

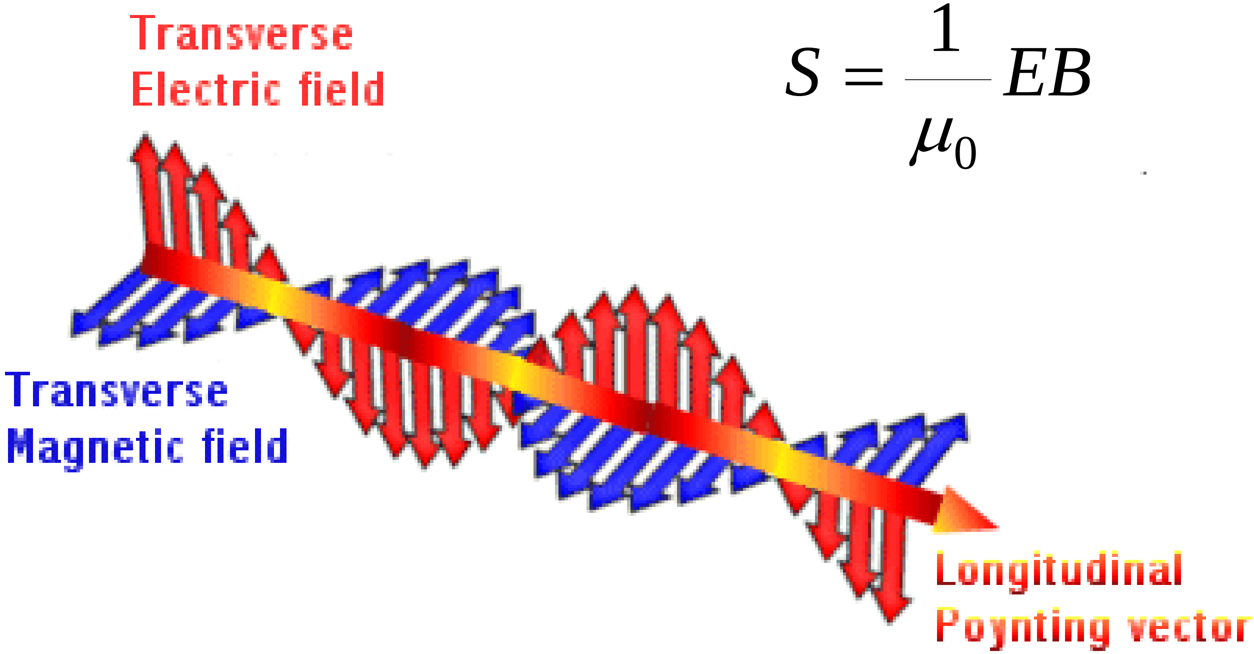
EM waves – Poynting vector

Poynting vector – has the direction of EM wave propagation and its size (amplitude) speaks about the rate of energy transport per unit area by the EM wave (unit: [W/m^2]):

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Due to the fact that vectors E and B are orthogonal (perpendicular), it is valid:

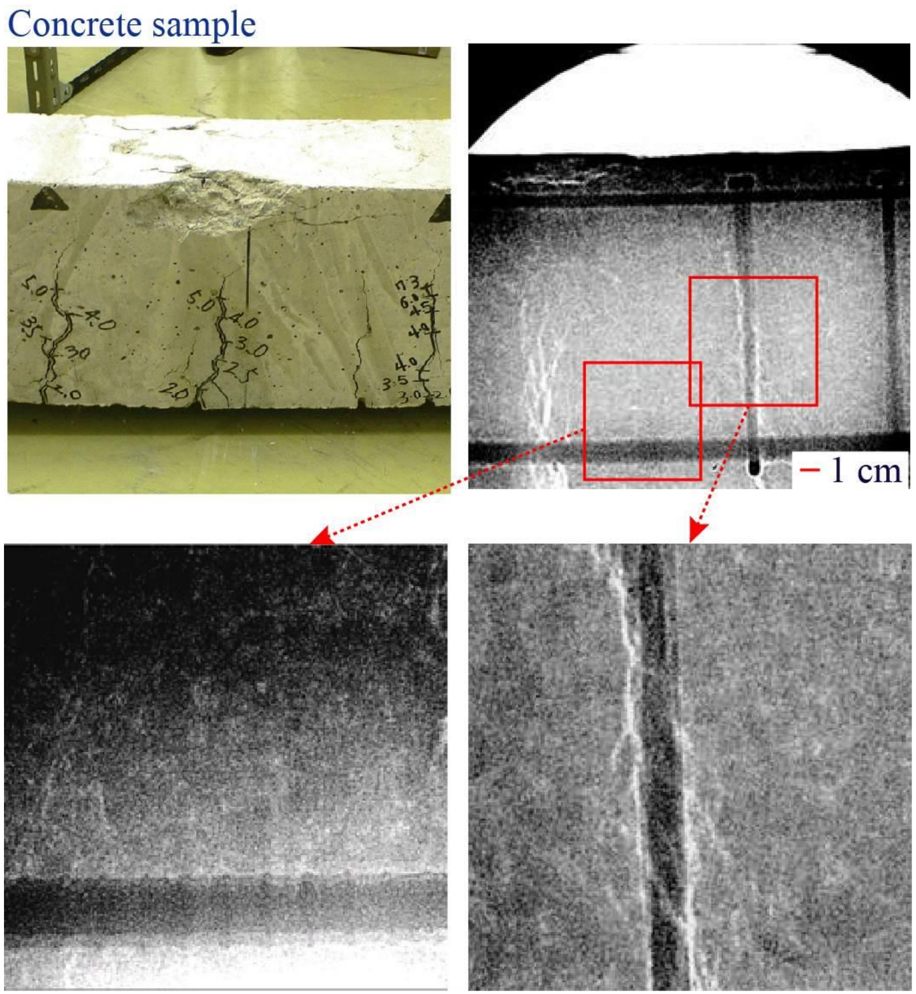
$$S = \frac{1}{\mu_0} EB$$



Named after its inventor John Henry Poynting.

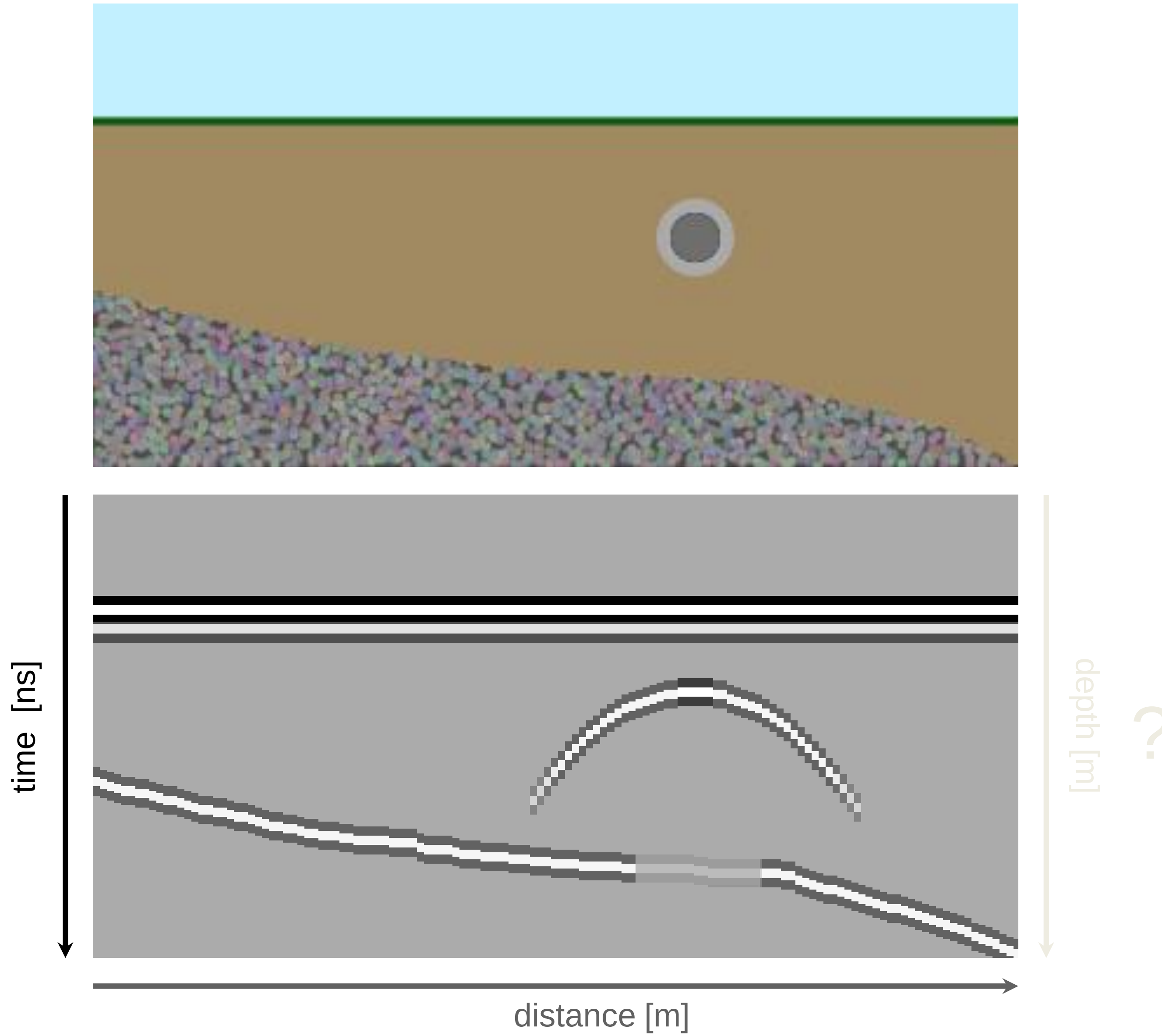
use of EM waves

in communication, science, medicine, engineering …



X-rays in materials inspection

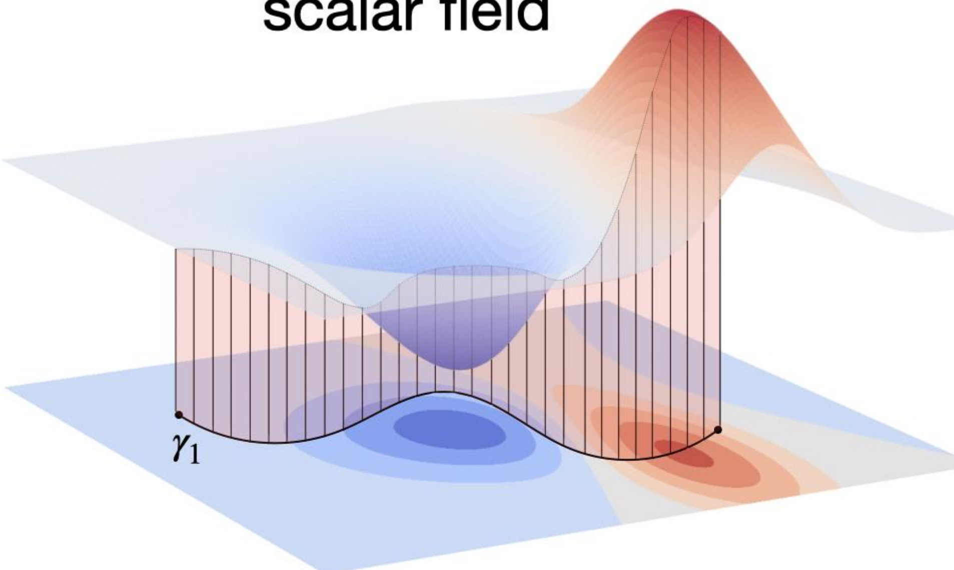
use of EM waves



GPR= Ground Penetrating Radar

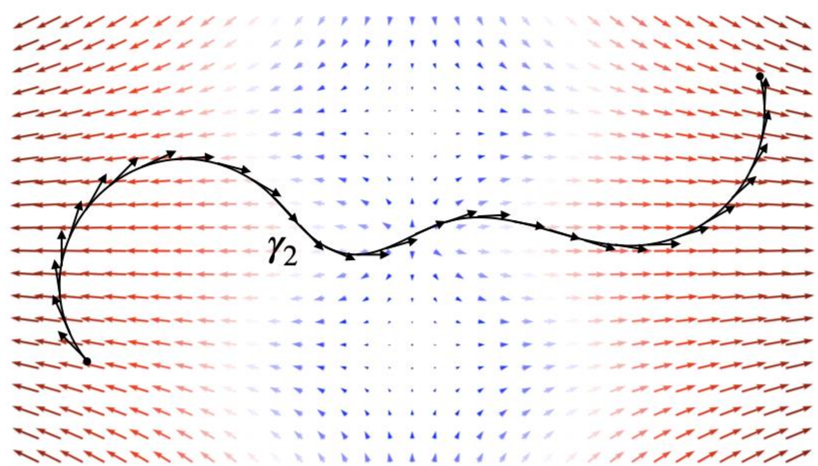
Appendix

Line integral of a scalar field



Area underneath the curve

Work of a vector field



Line integral of a vector field

Appendix – derivation of Gauss's law

Electric potential U of a charge Q (monopole):

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

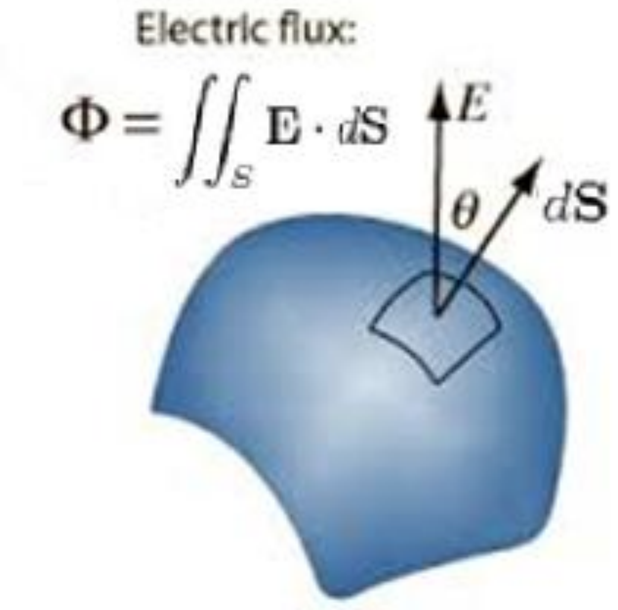
Electric field $\vec{E}$ of a charge Q (monopole):

$$\vec{E} = -gradU = -\nabla U = -\frac{\partial U}{\partial r}$$

$$\left|\vec{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{+Q}{r^2}$$

Electric flux is the total electric field $\vec{E}$ that crosses a given surface.
It is calculated by means of integration (e.g. closed surface integrals):

$$\Phi_E = \oiint_S \vec{E}\cdot d\vec{S}$$

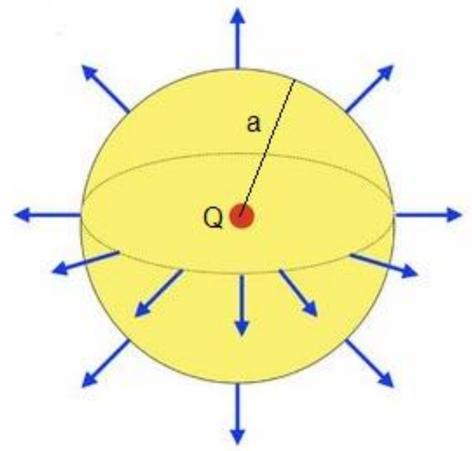


Appendix – derivation of Gauss's law

$$\left|\vec{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Electric flux in the case, when the close surface is a surace of a sphere and the monopole is in its centre (with radius *a*):

angle θ is always zero, because directions of *r* and *E* are identical



$$\Phi_E = \oint\!\!\oint_S \vec{E}\cdot d\vec{S} = \oint\!\!\oint_S \left|\vec{E}\right|\left|d\vec{S}\right|\cos\theta = \oint\!\!\oint_S \left|\vec{E}\right|\left|d\vec{S}\right| =$$

surface of a sphere is $4\pi a^2$

$$= \left|\vec{E}\right| \oint\!\!\oint_S \left|d\vec{S}\right| = \left|\vec{E}\right| 4\pi a^2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^2}4\pi a^2 = \frac{Q}{\varepsilon_0}$$

$$\Phi_E = \oint\!\!\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

Gauss's law for electric field

Situation is different in the case of magnetic field (magnetic induction $\vec{B}$), because magnetic field is a dipole field:

$$\Phi_B = \oint\!\!\oint_S \vec{B}\cdot d\vec{S} = 0$$

Gauss's law for magnetic field